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SELF-DUAL YANG-MILLS CONNECTIONS ON NON-SELF-DUAL 4-MANIFOLDS

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1. The principal results

The purpose of this article is to prove that self-dual Yang-Mills connections exist on a large class of four-dimensional Riemannian manifolds, specifically manifolds with no two dimensional anti-self-dual cohomology.

The differential geometric context is the following. We take M to be a compact connected oriented Riemannian 4-manifold, G a compact connected semi-simple Lie group, and P over M a principal G-bundle. The Yang-Mills functional is defined on the space of smooth connections $\mathcal{C}(P)$ on P as

$$\mathcal{YM}(A) = \frac{1}{2}\int_M |F_A|^2 = \frac{1}{2}\|F_A\|_{L_2}^2, \tag{1.1}$$

where F_A is the curvature of A, and (1.1) is a norm defined in terms of the Riemannian metric on M and the Cartan metric on the Lie algebra $\mathfrak{g}$ of G. This functional has been the subject of recent investigations by many authors in particular, [1], [11], [23].

The critical points of $\mathcal{YM}(\cdot)$ on $\mathcal{C}(P)$ are called Yang-Mills connections. A critical point $A \in \mathcal{C}(P)$ is distinguished by having harmonic curvature in the sense that

$$D_A^{\natural} F_A = 0 \qquad \text{(Yang-Mills equation)}, \tag{1.2a}$$

$$D_A F_A = 0 \qquad \text{(Bianchi identities)}, \tag{1.2b}$$

where D_A is the covariant exterior derivative.

Let $\hat{\mathfrak{g}} = P \times_{\mathrm{Ad}_G} \mathfrak{g}$ denote the vector bundle which is associated to P by the adjoint representation. The Hodge duality operator $*$ acts on sections of

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$\hat{\mathfrak{g}} \otimes \Lambda^p$, and defines an automorphism of $\hat{\mathfrak{g}} \otimes \Lambda^2$ with eigenvalues ± 1. Thus $\hat{\mathfrak{g}} \otimes \Lambda^2 = (\hat{\mathfrak{g}} \otimes P_+ \Lambda^2) \oplus (\hat{\mathfrak{g}} \otimes P_- \Lambda^2)$ where

$$P_\pm = \tfrac{1}{2}(1 \pm *). \tag{1.3}$$

The curvature $F_A \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^2)$, and if $P_- F_A = 0$ the connection is said to be self-dual while if $P_+ F_A = 0$, the connection is said to be anti-self-dual. If $A \in \mathcal{C}(P)$ is self-dual, then (1.2b) implies (1.2a); so every self-dual connection is a Yang-Mills connection. In fact, self-dual connections minimize $\mathcal{YM}(\cdot)$ over all $A \in \mathcal{C}(P)$.

For very basic reasons, a straightforward steepest descent method to find the global minima of $\mathcal{YM}(\cdot)$ is not successful in 4 dimensions, although this technique works in dimensions 2 and 3, [23]. In fact, the problem of finding critical points to $\mathcal{YM}(\cdot)$ in 4-dimensions is similar in many respects to the harmonic map problem in 2-dimensions and other conformally invariant variational problems. (See, for example, Uhlenbeck's review [24].) In the few cases where self-dual connections have been shown to exist, a high degree of symmetry in the base manifold has been exploited. This symmetry is manifested in the vanishing of the traceless, anti-self-dual Weyl tensor $\mathcal{W}_-$, which is part of Riemann curvature tensor. The Riemann curvature defines a self adjoint transformation

$$\mathcal{R}\colon \Lambda^2 \to \Lambda^2, \tag{1.4}$$

and $\mathcal{W}_-$ is the restriction of $\mathcal{R}$ to the traceless endomorphisms of $P_-\Lambda^2$. Atiyah, Hitchin and Singer [3] studied the properties of self-dual connections over base manifolds M which have positive scalar curvature and $\mathcal{W}_- = 0$ (self-dual spaces.). In this case, the bundle of projective anti-self-dual spinors PV_- has a complex structure and the following correspondence holds:

The Ward correspondence. Let E be a hermitian vector bundle with self-dual connection over a self-dual space M, and let $F = p^*E$ be the pulled back bundle. Then

1. F is holomorphic on PV_- with holomorphically trivial fibre.

2. There is a holomorphic isomorphism $\sigma\colon \tau^*\bar{F} \to F^*$, where $\tau\colon PV_- \to PV_-$ is the real structure, and σ induces a positive definite hermitian structure on the space of holomorphic sections of F on each fibre.

3. Every such bundle on PV_- is the pull-back of a bundle $E \to M$ with self-dual connection.

When $M = S^4$, the Ward correspondence has led to the construction of all self-dual connections on G-bundles over S^4, [2], [5], [13], [14], [15]. In this case, PV_- is naturally identified with PC^3, and algebraic techniques are used to construct the relevant complex structures [14], [15].

In this article self-dual connections are studied by analytic techniques, a result of which is that the self duality of the Riemannian curvature of the base manifold M is not required. Rather, we require that there be no anti-self-dual harmonic two-forms on M. That is,

$$P_{-}H^2_{\text{De Rham}}l(M) = 0, \tag{1.5}$$

where $H^2_{\text{De Rham}}(M)$ is the second cohomology group of the De Rham Complex: $0 \to \Gamma(\Lambda^0) \overset{d}{\to} \Gamma(\Lambda^1) \overset{d}{\to} \Gamma(\Lambda^2) \overset{d}{\to} \Gamma(\Lambda^3) \overset{d}{\to} \Lambda(\Lambda^4) \to 0$, and d is the exterior derivative. Our existence and classification results are Theorems 1.1, 1.2 and 1.3 below.

Theorem 1.1. *Let M be a compact oriented Riemannian manifold of dimension 4. Assume that $P_{-}H^2_{\text{De Rham}}(M) = 0$. Let G be a compact semi-simple Lie group. Then there exist principal G-bundles $P \to M$ which admit smooth irreducible self-dual connections.*

For G compact and semi-simple, principal G-bundles over M are classified up to isomorphism by the set of homotopy classes of maps from M into the classifying space for G, BG. (See the Appendix.) This set is denoted by $[M; BG]$, and there is a surjection

$$\phi: [M; BG] \to \mathbf{Z}^l \to 0, \tag{1.6}$$

where l is the number of nontrivial simple ideals which compose the Lie algebra of G. Let $P \to M$ be a principal G-bundle. Then the Pontrjagin classes $\{p_1{}^k(\hat{\mathfrak{g}})\}_{k=1}^l$ of the associated vector bundle $\hat{\mathfrak{g}} = P \times \mathrm{Ad}_G \mathfrak{g}$ specify the map ϕ. If G is simply connected, then ϕ is a bijection. If G is not simply connected, there is in addition, a map

$$\eta: [M; BG] \to H^2(M; \Pi_1(G)).$$

Now the map ϕ is a bijection on the kernel of the map η.

Theorem 1.2. *Assume as in Theorem 1.1 that M is a compact oriented 4-dimensional Riemannian manifold which satisfies $P_{-}H^2_{\text{De Rham}}(M) = 0$. Let G be a compact semi-simple Lie group. Let $P \to M$ be a principal G-bundle, all of whose Pontrjagin classes $\{P_1{}^k(\hat{\mathfrak{g}})\}_{k=1}^l$ are nonnegative. In addition, assume that the image of the isomorphism class of P under η in $H^2(M; \Pi_1(G))$ is trivial. Then the following statements are true*:

(i) *The space $\mathcal{C}(P)$ contains a smooth self-dual connection.*

(ii) *If the principal G-bundle over S^4 with the identical Pontrjagin classes admits an irreducible self-dual connection, then $\mathcal{C}(P)$ does also.*

(iii) *If M is a real analytic manifold, then there is a real analytic principal G-bundle P' which is isomorphic to P, and on which* (i) *and* (ii) *above are satisfied by real analytic connections.*

The conditions which make statement (ii) of Theorem 1.2 applicable have been determined by Atiyah, Hitchin and Singer [3]. These conditions are restated in Theorem 7.1.

We have nothing to say when the image under η of the isomorphism class of P in $H^2(M; \Pi_1(G))$ is nontrivial. It is possible that a combination of our techniques with the steepest descent techniques of Uhlenbeck [23] will yield results in these cases.

To count self-dual connections on P, we must take into account that the gauge group $\operatorname{Aut} P = \Gamma(P \times_{\operatorname{Ad}_G} G)$ has a natural action on $\mathcal{C}(P)$; cf. [11]. We denote this action by $(g, A) \to g(A)$ for $(g, A) \in \operatorname{Aut} P \times \mathcal{C}(P)$. The action respects both (1.2) and the condition of self-duality. For this reason, it is natural to consider the space of orbits in $\mathcal{C}(P)$ under the action of Aut P. The set of irreducible self-dual connections in $\mathcal{C}(P)$ modulo this action is called the space of moduli of self-dual connections in $\mathcal{C}(P)$. Atiyah, Hitchin and Singer proved that when M is a self-dual manifold, these moduli spaces are finite-dimensional manifolds. The generalization to those M where (1.5) holds is the next theorem.

Theorem 1.3. *Assume the conditions of Theorem* 1.2. *Suppose that* $P \to M$ *is a principal* G *bundle with* G *compact and semi-simple. Let* A *be a connection given by* (ii) *of Theorem* 1.2. *Then in a neighborhood of* A *in* $\mathcal{C}(P)/\operatorname{Aut} P$, *the space of moduli of irreducible self-dual connection is a manifold of dimension*

$$p_1(\hat{\mathfrak{g}}) - \tfrac{1}{2}(\dim G)(\chi - \tau), \tag{1.7}$$

where $p_1(\hat{\mathfrak{g}}) = \sum_{j=1}^{l} p_1{}^{l}(\hat{\mathfrak{g}})$ *is the sum of the* l *Pontrjagin classes of* $\hat{\mathfrak{g}}$, χ *is the Euler characteristic of* M, *and* τ *is the signature of* M.

Theorem 1.3 is a local result on the space of moduli. That is, there may be irreducible self-dual connections in $\mathcal{C}(P)$ for which the conclusions of the theorem do not hold. To state a stronger result we need assume more.

Theorem 1.4. *In addition to the assumptions on* M *and* P *in Theorem* 1.3, *assume that the following is true*: *The Riemannian metric tensor* g *on* M *is pointwise conformal to a metric* g' *on* M *whose curvature satisfies*

$$s' - 3w' > 0, \tag{1.8}$$

where $s'(x)$ *is the scalar curvature of* g', *and* $w'_-(x) = \sup_{\xi \in S^2 \subset \mathbf{R}^3} \mathcal{W}_-'^{ij}(x)\xi^i\xi^j$ *is the largest eigenvalue of the traceless anti-self-dual Weyl tensor of* g'. (*The metrics* g' *and* g *are pointwise conformal if* $g' = v^2(x)g$ *with* $v(x)$ *a smooth, strictly positive function on* M.) *Then the space of moduli of irreducible self-dual connections is globally a Hausdorff manifold of dimension given by* (1.7).

We remark that condition (1.5) implies that $\chi - \tau \leqslant 2$, so an immediate corollary of Theorems 1.3 and 1.4 is the following: Fix $p_1(\hat{\mathfrak{g}})$. Expression (1.7) is a function on the set {4-manifolds which satisfy the conditions of Theorem 1.4}. This function is minimized by the 4-sphere, because $\chi(S^4) = 2$ and $\tau(S^4) = 0$.

Using the Ward correspondence and Theorems 1.2 and 1.3, we have the following existence theorem for complex structures.

Theorem 1.5. *Let M be a 4-dimensional compact orientable Riemannian manifold with positive scalar curvature and $\mathcal{W}_- = 0$. Let $p: PV_- \to M$ be the bundle of projective anti-self-dual spinors. Let G be a compact semi-simple Lie group which has a unitary representation on a vector space L. There are holomorphic vector bundles F with fibre L over PV_- with the following properties*:

(1) *F is holomorphically trivial on each fibre.*

(2) *$\sigma{:}|\ \tau^*\bar{F} \to F^*$ is a holomorphic isomorphism.*

Theorems 1.1–1.4 imply that the self-dual connections on S^4 are stable with respect to all deformations of the standard Riemannian structure. In addition, (1.5) is satisfied on $S^3 \times S^1$, where the product metric satisfies (1.8), and on $P\mathbf{C}^2$, where the Fubini-Study metric satisfies (1.8). Therefore these spaces admit bundles with irreducible self-dual connections as given by the preceding theorems.

The question of whether irreducible self-dual connections exist when (1.5) is violated is not known in general. Thus for $S^2 \times S^2$ and the K^3 manifolds, we have no results. As an aside, we note that there are self-dual and anti-self-dual $SU(2)$ connections on $\mathbf{R}^2 \times S^2$, [21]. However we do prove the following approximation theorem. (See also Theorem 3.2.)

Theorem 1.6. *Let M be a compact oriented Riemannian 4-manifold with no assumption on its Riemannian curvature. Let G be a compact semi-simple Lie group. Let $P \to M$ be a principal G-bundle all of whose first Pontrjagin classes are nonnegative. In addition, assume that the isomorphism class of P has trivial image under η in $H^2(M; \Pi_1(G))$. Then given $\delta > 0$, there exists $A \in \mathcal{C}(P)$ with $\| P_-F_A \|_{L_2} < \delta$.*

As for anti-self-dual connections, note that reversing the orientation of the base manifold interchanges self-dual and anti-self-dual forms. Therefore Theorems 1.1–1.6 and the preceding discussion hold when self-dual P_-, $p_1(\hat{\mathfrak{g}})$, τ and $\mathcal{W}_-$ are replaced by anti-self-dual P_+, $-p_1(\hat{\mathfrak{g}})$, $-\tau$ and $\mathcal{W}_+$, respectively.

The remainder of this article contains the proofs of the preceding theorems, and it is organized in the following way. The proofs require three crucial technical theorems, Theorems 3.2, 6.1 and 8.2. §2 establishes our notation and convention and §§3–6 contain the proofs of Theorems 3.2 and 6.1. Theorem 8.2 is a generalization of Theorem 1.6, and the proof requires §§7 and 8. §7 is a review of certain facts about self-dual connections on S^4 and $\mathbf{R}^4$. These facts are used in §8 to complete the proof of Theorem 8.2. Finally, in §9, are the proofs of Theorems 1.2–1.4 completed. The appendix is a review of characteristic classes and the classification of principal bundles on 4-manifolds.

2. Notation

The purpose of this section is to establish our notation. Let $P \to M$ be a principal G-bundle where G is a compact semi-simple Lie group. The p'th exterior power of the cotangent bundle Λ^p is a vector bundle over M with structure group $SO(4)$. Hence a connection $A \in \mathcal{C}(P)$ and the Riemannian connection on Λ^p define the covariant derivative

$$\nabla_A\colon \Lambda(\hat{\mathfrak{g}} \otimes \Lambda^p) \to \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^p \otimes \Lambda^1). \tag{2.1}$$

The natural projection $\alpha\colon \Lambda^p \otimes \Lambda^1 \to \Lambda^{p+1}$ allows one to define the exterior derivative $D_A\colon \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^p) \to \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^{p+1})$ as $D_A = \alpha \circ \nabla_A$. The curvature F_A of the connection A is a section $\hat{\mathfrak{g}} \otimes \Lambda^2$; its relation to D_A is

$$D_A D_A \sigma = F_A \wedge \sigma - \sigma \wedge F_A \tag{2.2}$$

for $\sigma \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^p)$. The full curvature of ∇_A is a direct sum of F_A and the Riemannian curvature of M. The Riemannian curvature is the $SO(4)$ Lie algebra valued 2-form

$$\mathcal{R} = \tfrac{1}{4}\mathcal{R}_{\mu\nu\alpha\beta}\omega^\mu \wedge \omega^\nu \oplus \omega^\alpha \wedge \omega^\beta, \tag{2.3}$$

where $\{\omega^\nu\}_{\nu=1}^4$ is a local orthonormal frame for Λ^1, and we have identified the Lie algebra $\mathcal{SO}(4)$ with Λ^2. The Lie algebra $\mathcal{SO}(4) \approx \mathcal{SO}(3) \oplus \mathcal{SO}(3)$, and this corresponds to the identity $\Lambda^2 = P_+\Lambda^2 \oplus P_-\Lambda^2$.

Let $\{x_\pm^i\}_{i=1}^3$ be a local orthonormal basis for $P_\pm\Lambda^2$, respectively. Then $\mathcal{R}$ has the decomposition

$$\begin{aligned} \mathcal{R} = {} & \mathcal{W}_+^{ij} x_+^i \otimes x_+{}^j + \mathcal{W}_-^{ij} x_-^i \otimes x_-^j + \mathcal{B}^{ij} x_+^i \otimes x_-^j \\ & + \mathcal{B}^{Tij} x_-^i \otimes x_+^j + \frac{s}{6}\left(x_+^j \otimes x_+^j + x_-^j \otimes x_-^j\right), \end{aligned} \tag{2.4}$$

where $\mathcal{W}_\pm^{ij}$ are traceless, and are respectively the self-dual and anti-self-dual parts of the Weyl tensor. The function s on M is the scalar curvature, and $\mathcal{B}^{ij}$ is the traceless Ricci tensor. The above representation decomposes $\mathcal{R}$ into its irreducible components with respect to $SO(4)$ (See [10] for more details.)

The Riemannian metric and the Cartan form on $\mathfrak{g}$ give $\Gamma(\hat{\mathfrak{g}} \otimes \Lambda^p)$ a pointwise inner product $(\cdot\,,\cdot)$, the L_2 inner product $\langle\cdot\,,\cdot\rangle_{L_2}$, and the L_p norms

$$\|\omega\|_{L_p} = \left(\int_M * (\omega, \omega)^{p/2}\right)^{1/p}. \tag{2.5}$$

It is with respect to the L_2 inner product that the adjoints $\nabla_A^\natural$ and $D_A^\natural$ are defined. Of particular interest are these operators and their Laplacians on $\Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ and $\Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$.

Definition 2.1. The operator $\mathfrak{D}_A: \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1) \to \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ on $a \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$ is

(2.6) $$\mathfrak{D}_A a = P_- D_A a;$$

its formal adjoint on $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ is

(2.7) $$\mathfrak{D}_A^\natural u = {}^*D_A u.$$

Propositioin 2.2. *Let $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$. Then with respect to a local orthonormal frame $\{x_-^i\}_{i=1}^3$ of $P_-\Lambda^2$,*

(2.8) $$(\mathfrak{D}_A \mathfrak{D}_A^+ u) = \tfrac{1}{2}\left\{(\nabla_A^+ \nabla_A u)^i + \sqrt{2}\,\varepsilon^{ijk}\left[P_-F_A^k, u^j\right] + \tfrac{2}{3} s u^i - 2\mathfrak{W}_-^{ij} u^j\right\},$$

where $P_-F_A^k = (x_-^k, P_-F_A)$.

Proposition 2.3. *Let $a \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$. Then with respect to a local orthonormal frame $\{\omega^\nu\}_{\nu=1}^4$ of Λ^1,*

(2.9) $$\left(2\mathfrak{D}_A^\natural \mathfrak{D}_A a + \nabla_A \nabla_A^\natural a\right)_\alpha = \left(\nabla_A^\natural \nabla_A a\right)_\alpha - 2\left[P_+ F_{\alpha\beta}, a_\beta\right] + a_\rho \mathfrak{R}_{\rho\beta\alpha\beta},$$

where ∇_A^+ is the adjoint of $\nabla_A: \Gamma(\hat{\mathfrak{g}}) \to \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$. (2.8) *and* (2.9) *are known as Bochner-Weizenböch formulas; cf.* [11], [9].

3. The self duality equations

As in §2, G is a compact semi-simple Lie group, and $P \to M$ is a principal G-bundle. Let $A_0 \in \mathcal{C}(P)$ be fixed. Because $\mathcal{C}(P)$ is an affine space, any connection $A \in \mathcal{C}(P)$ can be written uniquely as

(3.1) $$A = A_0 + a \quad \text{with } a \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1).$$

Therefore if $A \in \mathcal{C}(P)$ has self-dual curvature, then

(3.2) $$0 = P_-F_{A_0} + \mathfrak{D}_{A_0} a + a\#a,$$

where we have defined

(3.3) $$a\#b = \tfrac{1}{2} P_-(a \wedge b + b \wedge a).$$

Conversely, if $a \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$ satisfies (3.2), then $A = A_0 + a \in \mathcal{C}(P)$ has self-dual curvature. In order to find a self-dual connection, it is sufficient to find $A_0 \in \mathcal{C}(P)$ such that (3.2) has a smooth solution in $\Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$.

Because the operator $\mathfrak{D}_{A_0}$ is not properly elliptic, it is convenient to write $a = \mathfrak{D}_{A_0}^\natural u$ for $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ and replace (3.2) by

(3.4) $$\mathfrak{D}_{A_0} \mathfrak{D}_{A_0}^\natural u + \mathfrak{D}_{A_0}^\natural u \# \mathfrak{D}_{A_0}^\natural u = -P_-F_{A_0}.$$

(3.4) is a properly elliptic system. Notice that if A_0 is itself self-dual, then (3.4) automatically has a solution, namely $u \equiv 0$. If F_{A_0} is small in an appropriate norm, but nonzero, it is still reasonable to assume that (3.4) has a solution $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ which is also small. This is the case, and the proof of Theorem 1.2 requires the construction of an implicit function theorem for (3.4). A similar technique was used successfully to prove the existence of static solutions to the Yang-Mills-Higgs equations on $\mathbf{R}^3$ (see [16, Chapter IV], [22]).

The operator $\mathfrak{D}_{A_0}\mathfrak{D}_{A_0}^{\natural}$ is an elliptic self-adjoint operator on the space of square integrable sections of $\hat{\mathfrak{g}} \otimes P_-\Lambda^2$. It is a standard result that the spectrum of $\mathfrak{D}_{A_0}\mathfrak{D}_{A_0}^{\natural}$ is discrete, and the lowest eigenvalue is nonnegative.

Definition 3.1. For $A \in \mathcal{C}(P)$, define

$$\mu(A) \equiv \text{lowest eigenvalue of } \mathfrak{D}_A\mathfrak{D}_A^{\natural}.$$

If $\mu(A) > 0$, define

$$\zeta(A) \equiv \mu(A)^{-1/2}\left(1 + \mu(A) + \|P_-F_A\|_{L_3}^3\right)^{-1/2}, \tag{3.5a}$$

$$\delta(A) \equiv \|P_-F_A\|_{L_2} + \zeta(A)\|P_-F_A\|_{L_{4/3}}\left(1 + \|F_A\|_{L_4}\right). \tag{3.5b}$$

If $\mu(A) = 0$, define $\zeta(A) = \delta(A) = +\infty$.

The basic existence theorem is

Theorem 3.2. *Let M be a four-dimensional compact oriented Riemannian manifold. Let $P \to M$ be a principal G-bundle with G a compact semi-simple Lie group. There exists $\varepsilon_0 > 0$ which is independent of $A_0 \in \mathcal{C}(P)$ and P with the following significance: If*

$$\delta(A_0) < \varepsilon_0, \tag{3.6}$$

then there exists a solution $a \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$ to (3.2). In fact, $a = \mathfrak{D}_{A_0}^{\natural}u$ where $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ is a solution to (3.4). Further, there exists a constant $c < \infty$ which is independent of $A_0 \in \mathcal{C}(P)$ and P such that

$$\left\langle \nabla_{A_0}a, \nabla_{A_0}a \right\rangle_{L_2} + \langle a, a\rangle_{L_2} \leqslant c\delta(A_0). \tag{3.7}$$

Corollary 3.3. *The connection $A = A_0 + a \in \mathcal{C}(P)$ is self-dual.*

4. An L_p threshold for self duality

The proof of Theorem 3.2 comprises this and the next section. The solution u to (3.4) will be given by a convergent expansion

$$u = \sum_{n=1}^{\infty} u_n. \tag{4.1}$$

The expansion parameter is $\delta(A_0)$. Each term u_n in this expansion is a solution to a linear equation of the form

$$\mathfrak{D}_{A_0}\mathfrak{D}_{A_0}^{\natural}v = q \tag{4.2}$$

for $v \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$. The relevant properties of a solution v to (4.2) are summarized in the following theorem which is proved in §5.

Theorem 4.1. *Let P and M be as in Theorem 3.2. Let $A_0 \in \mathcal{C}(P)$ and suppose that $\mu(A_0) > 0$. Let $q \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$. Then there exists a unique C^∞ solution v to (4.2) such that*

$$\|\mathfrak{D}_{A_0}^{\natural}v\|_{L_2} \leq C_1\zeta(A_0)^{-1}\|q\|_{L_{4/3}}, \tag{4.3}$$

$$\left(\|\nabla_{A_0}(\mathfrak{D}_{A_0}^{\natural}v)\|_{L_2}^2 + \|\mathfrak{D}_{A_0}^{\natural}v\|_{L_2}^2\right)^{1/2} \leq C_1\left\{\|q\|_{L_2} + \zeta(A_0)^{-1}\|q\|_{L_{4/3}}\left(1 + \|F_{A_0}\|_{L_4}\right)\right\}, \tag{4.4}$$

$$\|\mathfrak{D}_{A_0}^{\natural}v\|_{L_4} \leq C_1\left\{\|q\|_{L_2} + \zeta(A_0)^{-1}\|q\|_{L_{4/3}}\left(1 + \|F_{A_0}\|_{L_4}\right)\right\}. \tag{4.5}$$

The constant C_1 is independent of P, $A_0 \in \mathcal{C}(P)$ and q.

Proof of Theorem 3.2 *assuming Theorem* 4.1. The proof uses an iterative method for solving a quadratic equation.

The formal aspects of the proof are the following. Each u_k in the sum (4.1) is the solution to the linear equation

$$\mathfrak{D}_{A_0}\mathfrak{D}_{A_0}^{\natural}u_k = q_k, \tag{4.6}$$

where

$$q_1 = -P_-F_{A_0}, \tag{4.7}$$

and for $k > 1$

$$q_k - 2\sum_{j=1}^{k-2} \mathfrak{D}_{A_0}^{\natural}u_j \# \mathfrak{D}_{A_0}^{\natural}u_{k-1} - \mathfrak{D}_{A_0}^{\natural}u_{k-1} \# \mathfrak{D}_{A_0}^{\natural}u_{k-1}. \tag{4.8}$$

Assuming each u_k exists, define the partial sums

$$s_m = \sum_{k=1}^{m} u_k. \tag{4.9}$$

Then as a consequence of (4.6)–(4.9) we have

$$P_-F_{A_0} + \mathfrak{D}_{A_0}\mathfrak{D}_{A_0}^{\natural}s_m + \mathfrak{D}_{A_0}^{\natural}s_{m-1} \# \mathfrak{D}_{A_0}^{\natural}s_{m-1} = 0. \tag{4.10}$$

Hence if the $\lim_{m\to\infty} s_m = u$ exists in the appropriate sense (cf. Lemmas 4.5 and 4.7), then u is a solution to (3.4), and $a = \mathfrak{D}_{A_0}^{\natural}u$ is a solution to (3.2).

We now use Theorem 4.1 to justify the preceding analysis. The proof of Theorem 3.2 is considerably simplified by introducing Hilbert spaces of sections of $\hat{\mathfrak{g}} \otimes \Lambda^p$.

Definition 4.2. For $u, v \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^p)$, define

$$(4.11) \qquad \begin{aligned} \langle u, v\rangle_H &= \langle \nabla_{A_0} u, \nabla_{A_0} v\rangle_{L_2} + \langle u, v\rangle_{L_2}, \\ \|u\|_H &= \langle u, u\rangle_H^{1/2}. \end{aligned}$$

Definition 4.3. The Hilbert spaces $\mathcal{H} = \mathcal{H}(A_0)$ and $\mathcal{K} = \mathcal{K}(A_0)$ are the completions of $\Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ and $\Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$, respectively, in the norm $\|\cdot\|_H$.

The space $\mathcal{H}$ depends on the choice of $A_0 \in \mathcal{C}(P)$. This should be kept in mind. Technically any two $\mathcal{H}(A_0)$ and $\mathcal{H}(A_1)$ are isomorphic, but not isometric. A similar remark is true for $\mathcal{K}$.

Proposition 4.4. *Let ε_0 in* (3.6) *satisfy*

$$\varepsilon_0 \leqslant (32C_1)^{-1},$$

with C_1 given in Theorem 4.1. *Then each u_k, q_k exists and is C^∞. Further for each $k \geqslant 1$ we have*

$$(4.12) \qquad \|\mathfrak{D}_{A_0}^{\natural} u_k\|_{L_2} \leqslant \frac{1}{16C_1}\left(16C_1^2\delta(A_0)\right)^k\left(1 + \|F_{A_0}\|_{L_4}\right)^{-1},$$

$$(4.13) \qquad \begin{aligned} \|\mathfrak{D}_{A_0}^{\natural} u_k\|_H &\leqslant \frac{1}{16C_1}\left(16C_1^2\delta(A_0)\right)^k, \\ \|\mathfrak{D}_{A_0}^{\natural} u_k\|_{L_4} &\leqslant \frac{1}{16C_1}\left(16C_1^2\delta(A_0)\right)^k. \end{aligned}$$

Proof. The proof is by induction on the integer k. The induction begins with $k = 1$. Then $q_1 = -P_-F_0$. Since $\delta(A_0) < \varepsilon_0$, Theorem 4.1 states that there exists a unique $u_1 \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ which satisfies

$$(4.14) \qquad \mathfrak{D}_{A_0}\mathfrak{D}_{A_0}^{\natural} u_1 = q_1 = -P_-F_0.$$

(4.12) and (4.13) follows from (4.3)–(4.5) and the definition of $\delta(A_0)$.

The induction proof is completed by demonstrating that if (4.12) and (4.13) are satisfied for $j < k$, then they are satisfied for $j = k$. Indeed, since q_k depends on the functions $\{u_j;\ j \leqslant k-1\}$, we have

$$(4.15) \qquad \begin{aligned} \|q_k\|_{L_1} &\leqslant 4 \sum_{j=1}^{k-1} \|\mathfrak{D}_{A_0}^{\natural} u_j\|_{L_2} \|\mathfrak{D}_{A_0}^{\natural} u_{k-1}\|_{L_2}, \\ \|q_k\|_{L_2} &\leqslant 4 \sum_{j=1}^{k-1} \|\mathfrak{D}_{A_0}^{\natural} u_j\|_{L_4} \|\mathfrak{D}_{A_0}^{\natural} u_{k-1}\|_{L_4}. \end{aligned}$$

It follows from the hypothesis on u_j for $j < k$ that

$$\begin{aligned} \|q_k\|_{L_1} &\leq 4\left(\frac{1}{16C_1}\right)^2 \left(16C_1^2\delta(A_0)\right)^k \left(1 + \|F_{A_0}\|_{L_4}\right)^{-2} \sum_{j=1}^{k-2} \left(16C_1^2\delta(A_0)\right)^j \\ &\leq 8\left(\frac{1}{16C_1}\right)^2 \left(16C_1^2\delta(A_0)\right)^k \left(1 + \|F_{A_0}\|_{L_4}\right)^{-2}, \end{aligned} \tag{4.16}$$

and that

$$\|q_k\|_{L_2} \leq 8\left(\frac{1}{16C_1}\right)^2 \left(16C_1^2\delta(A_0)\right)^k. \tag{4.17}$$

In the above analysis, it has been assumed that $\varepsilon_0 \leq (32C_1^2)^{-1}$. Holder's inequality gives

$$\|q_k\|_{L_{4/3}} \leq \|q_k\|_{L_1}^{1/2} \|q_k\|_{L_2}^{1/2}. \tag{4.18}$$

Thus $q_k \in L_2 \cap L_{4/3} \cap \Gamma^2(\hat{\mathfrak{g}} \otimes P_{-}\Lambda^2)$. Theorem 4.1 states that u_k exists and is an element of $\Gamma^2(\hat{\mathfrak{g}} \otimes P_{-}\Lambda^2)$. Finally (4.3)–(4.5) and (4.16)–(4.18) give

$$\begin{aligned} \|\mathfrak{D}_{A_0}^{\natural} u_k\|_{L_2} &\leq 8C_1\left(\frac{1}{16C_1}\right)^2 \left(16C_1^2\delta(A_0)\right)^k \left(1 + \|F_{A_0}\|_{L_4}\right)^{-1}, \\ \|\mathfrak{D}_{A_0}^{\natural} u_k\|_{H} &\leq 16C_1\left(\frac{1}{16C_1^2}\right) \left(16C_1^2\delta(A_0)\right)^k, \\ \|\mathfrak{D}_{A_0}^{\natural} u_k\|_{L_4} &\leq 16C_1\left(\frac{1}{16C_1^2}\right) \left(16C_1^2\delta(A_0)\right)^k. \end{aligned} \tag{4.19}$$

Making the cancellations in (4.19) proves that the induction hypothesis is satisfied for u_k as claimed.

We now prove that the conditions of Proposition 4.4 ensure the convergence of the partial sums s_m and $\mathfrak{D}_{A_0}^{\natural} s_m$ to a limit which satisfies (3.4).

Lemma 4.5. *Let A_0 satisfy the conditions set forth in Proposition 4.4. Then the sequence $\{s_m\}_{m=1}^{\infty}$ defined by (4.9) converges to a limit $u \in \mathcal{H}(A_0)$, and the sequence $\{\mathfrak{D}_{A_0}^{\natural} s_m\}_{m=1}^{\infty}$ to a limit $a \in \mathcal{H}(A_0)$. Further*

$$\mathfrak{D}_{A_0}^{\natural} u = a. \tag{4.20}$$

In order to prove this lemma, a technical result is needed. The proof is deferred until §5.

Lemma 4.6. *There exists a constant $0 < C_2 < \infty$ which is independent of $A_0 \in \mathcal{C}(P)$ and P with the following significance: If $\mu(A_0) > 0$, then*

$$C_2\zeta(A_0)\|v\|_H \leq \|\mathfrak{D}_{A_0}^{\natural} v\|_{L_2} \leq \frac{1}{C_2}\|v\|_H \tag{4.21}$$

for all $v \in \mathcal{H}(A_0)$.

Proof of Lemma 4.5. To prove the convergence of $\{s_m\}$ and $\{\mathfrak{D}^{\natural}_{A_0} s_m\}$ we show that these sequences are Cauchy. Indeed, from Proposition 4.4 one has for all $n, m > N$

$$\|s_n - s_m\|_H \leqslant \zeta^{-1}(A_0) C_2^{-1} \|\mathfrak{D}^{\natural}_{A_0}(s_n - s_m)\|_{L_2} \leqslant \left(8\zeta(A_0) C_2 C_1\right)^{-1} \cdot 2^{-N},$$

$$\|\mathfrak{D}^{\natural}_{A_0} s_n - \mathfrak{D}^{\natural}_{A_0} s_m\|_H \leqslant \frac{1}{8C_1} 2^{-N}.$$

Thus both sequences are Cauchy. (4.20) is a standard result.

Lemma 4.7. *The functions*

$$v_m = \mathfrak{D}_{A_0} \mathfrak{D}^{\natural}_{A_0} s_m + \mathfrak{D}^{\natural}_{A_0} s_m \# \mathfrak{D}^{\natural}_{A_0} s_m - P_- F_0 \tag{4.22}$$

converge to zero in L_2.

Proof. A calculation based on the fact that $\mathfrak{D}_{A_0} = P_- \mathfrak{D}_{A_0}$ gives

$$\|\mathfrak{D}_{A_0} b\|_{L_2} \leqslant 8 \|b\|_H, \tag{4.23}$$

for all $b \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$. Let $n, m > N$. Then

$$\begin{aligned} \|v_n - v_m\|_{L_2} \leqslant\ & 8\|\mathfrak{D}^{\natural}_{A_0} s_n - \mathfrak{D}^{\natural}_{A_0} s_m\|_H \\ & + \|\mathfrak{D}^{\natural}_{A_0}(s_m - s_n) \# \mathfrak{D}^{\natural}_{A_0}(s_m + s_n)\|_{L_2}, \end{aligned} \tag{4.24}$$

where we have used the fact that the # operator is symmetric. Holder's inequality and Proposition 4.4 yield

$$\|v_n - v_m\|_{L_2} \leqslant O(2^{-N}). \tag{4.25}$$

Thus the sequence $\{v_n\}$ is Cauchy. Using (4.10) and Proposition 4.4 one can show similarly that the strong limit of the sequence $\{v_n\}$ is zero as claimed.

Proof of Theorem 3.2 (*completion*). Since $v_n \to 0$ in L_2, $u = \lim_{m \to \infty} s_m$ is a weak solution to (3.4) in the following sense: For all $v \in L_2(\hat{\mathfrak{g}} \otimes P_- \Lambda^2)$,

$$\langle v, P_- F_{A_0} + \mathfrak{D}_{A_0} \mathfrak{D}^{\natural}_{A_0} u + \mathfrak{D}^{\natural}_{A_0} u \# \mathfrak{D}^{\natural}_{A_0} u \rangle_{L_2} = 0. \tag{4.26}$$

Since A_0 is smooth, $u \in L^2_2(\hat{\mathfrak{g}} \otimes P_- \Lambda^2)$ (cf. [9] for definitions). The claim that $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_- \Lambda^2)$ follows from standard L_p estimates for elliptic systems. We omit the proof (cf. [18, Chapters 5 and 6]). (3.7) follows by summing (4.12) and (4.13).

5. The linearized equation

For fixed $A_0 \in \mathcal{C}(P)$, we study the properties of the equation

$$\mathfrak{D}_{A_0} \mathfrak{D}^{\natural}_{A_0} u = q, \tag{5.1}$$

and prove Theorem 4.1 and Lemma 4.6. This will complete the proof of Theorem 3.2.

The solution u to (5.1) is formally a critical point of the functional

$$S_q[u] = \frac{1}{4}\langle \nabla_{A_0}u, \nabla_{A_0}u\rangle_{L_2} + \frac{\sqrt{2}}{4}\langle u, PF_{A_0}(u)\rangle_{L_2} + \frac{1}{6}\langle u, su\rangle_{L_2} - \frac{1}{2}\langle u, \mathcal{W}_-(u)\rangle_{L_2} - \langle u, q\rangle_{L_2}. \tag{5.2}$$

In terms of local orthonormal basis for $\hat{\mathfrak{g}} \otimes P_-\Lambda^2$,

$$\begin{aligned} \left(P_-F_{A_0}(u)\right)^i &= \varepsilon^{ijk}\left[P_-F_{A_0}{}^k, u^j\right], \\ (su)^i &= su^i, \\ \left(\mathcal{W}_-(u)\right)^i &= \mathcal{W}_-{}^{ij}u^j. \end{aligned} \tag{5.3}$$

Clearly $S_q[u]$ is finite for $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$, and for such u we have

$$S_q[u] = \tfrac{1}{2}\left\langle \mathcal{D}^\natural_{A_0}u, \mathcal{D}^\natural_{A_0}u\right\rangle - \langle q, u\rangle_{L_2}. \tag{5.4}$$

Proposition 5.1. *Let M and P be as in Theorem* 3.2. *Let $A_0 \in \mathcal{C}(P)$. Suppose that $\mu(A_0) > 0$ and $q \in L_{4/3}$. Then there is a unique weak solution $u \in \mathcal{H}$ to* (5.1) *in the sense that for all $v \in \mathcal{H}$,*

$$\left\langle \mathcal{D}^\natural_{A_0}v, \mathcal{D}^\natural_{A_0}u\right\rangle_{L_2} - \langle v, q\rangle_{L_2} = 0. \tag{5.5}$$

If q is C^∞, then u is C^∞ and (5.1) *is satisfied pointwise.*

Proof. The Proposition is proved by using the calculus of variations. We begin by establishing an important property of the Banach space $\mathcal{H}$, namely that $\mathcal{H}$ imbeds in L_4 with imbedding constant independent of A_0 and P. With this fact established, it is straightforward to show that the functional $S_q\{\cdot\}$ can be defined on $\mathcal{H}$ by representing elements in $\mathcal{H}$ by Cauchy sequences in $\Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ with respect to the H-norm. Lemma 4.6 will follow immediately also. Lemma 4.6 implies that $S_q[\cdot]$ is a strictly convex functional which satisfies a coercive lower bound. An additional technical lemma concerning the strict-convexity of $S_q[\cdot]$ and its differentiability is needed to apply known results from the calculus of variations. After appealing to these results, Proposition 5.1 will follow. We now present the details.

Lemma 5.2. *Let $u \in \mathcal{H}$ (or $\mathcal{H}$). Then $|u|$ is an L^1_2 function and*

$$\|\,|u|\,\|_{L^1_2} \leq \|u\|_H. \tag{5.6}$$

In addition, there exists a constant C_4 which is independent of $A_0 \in \mathcal{C}(P)$ and P such that for all $u \in \mathcal{H}$ (or $\mathcal{H}$)

$$\|u\|_{L_4} \leq C_4\|u\|_H. \tag{5.7}$$

Proof. Recall that the L_2^1 norm on functions [19] is

$$\|f\|_{L_2^1} = \left(\langle df, df\rangle_{L_2} + \langle f, f\rangle_{L_2}\right)^{1/2}, \tag{5.8}$$

for $f \in \Gamma(M)$. The first statement of the lemma and (5.6) is Kato's inequality [16, Chapter IV]. (5.7) follows from (5.6) and a Sobolev inequality [19].

Lemma 5.3. *If $q \in L_{4/3}$, then $S_q[\cdot]$ extends to a finite functional on $\mathcal{H}$. In addition* (5.4) *holds for all $u \in \mathcal{H}$.*

Proof. We remark that the right-hand sides of both (5.4) and (5.2) define strongly continuous functional on $\mathcal{H}$. In fact for $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$, the right-hand side of (5.2) is bounded by

$$\begin{aligned} &\leqslant \tfrac{1}{4}\|u\|_H^2 + 2\|u\|_{L_4}^2 \|P_-F_{A_0}\|_{L_2} + \tfrac{1}{2}\|\tfrac{1}{3}s \\ &\quad + |\mathcal{W}_-|\,\|_{L_\infty}\|u\|_H^2 + \|q\|_{L_{4/3}}\|u\|_{L_4} \\ &\leqslant \text{const.}\left(1 + \|P_-F_{A_0}\|_{L_2}\right)\|u\|_H^2 + C_4^2\|q\|_{L_{4/3}}^2, \end{aligned} \tag{5.9}$$

where we have used Hölder's inequality and (5.7). Meanwhile, for $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$, the left-hand side of (5.4) is bounded by

$$\begin{aligned} &\leqslant 8\|\nabla_{A_0}u\|_{L_2}^2 + \|q\|_{L_{4/3}}\|u\|_{L_4} \\ &\leqslant 10\|u\|_H^2 + C_4^2\|q\|_{L_{4/3}}^2. \end{aligned} \tag{5.10}$$

The extension of $S_q[\cdot]$ to $\mathcal{H}$ and the equality of (5.2) and (5.4) follow from (5.9) and (5.10) by representing an arbitrary $u \in \mathcal{H}$ as a limit of sequences in $\Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ which are Cauchy sequences with respect to the norm $\|\cdot\|_H$ on $\Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$.

Lemma 5.4. *There is a constant $C_2 > 0$ which depends only on the Riemannian structure of M with the following significance: If $\mu(A_0) > 0$, then for all $u \in \mathcal{H}$ and $q \in L_{4/3}$,*

$$S_q[u] \geqslant [C_2\zeta(A_0)]^2\|u\|_H^2 - [C_2\zeta(A_0)]^{-2}\|q\|_{L_{4/3}}^2. \tag{5.11}$$

(*Recall that $\zeta(A_0)$ is defined in* (3.5).)

Proof. It is enough to establish (5.11) for $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$. Using (5.4) and (5.2) respectively, we obtain the estimates

$$S_q[u] \geqslant \mu(A_0)\|u\|_{L_2}^2 - \langle q, u\rangle_{L_2}, \tag{5.12a}$$

$$\begin{aligned} S_q[u] \geqslant \|\nabla_{A_0}u\|_{L_2}^2 &- 8\|P_-F_{A_0}\|_{L_3}\|u\|_{L_2}^{2/3}\|u\|_{L_4}^{4/3} \\ &- C(M)\|u\|_{L_2}^2 - \langle q, u\rangle_{L_2}. \end{aligned} \tag{5.12b}$$

In (5.12b) we have used Hölder's inequality. The constant $C(M)$depends only on the Riemannian structure of M. (5.7) is used to estimate the $\|u\|_{L_4}$ terms;

thus from (5.12b) we have

$$(5.13)\quad S_q[u] \geq \frac{1}{8}\|u\|_H^2 - \|u\|_{L_2}^2 C(M)\left(1 + \|P_-F_{A_0}\|_{L_3}^3\right) - \langle q, u\rangle_{L_2},$$

where $C(M)$ is a (different) constant which depends only on the Riemannian structure of M. (5.12a) is used to bound $\|u\|_{L_2}^2$. The resulting inequality is

$$(5.14)\quad \left(1 + C(M)\mu(A_0)^{-1}\left(1 + \|P_-F_{A_0}\|_{L_3}^3\right)\right)\left(S_q[u] + \langle q, u\rangle_{L_2}\right) \geq \tfrac{1}{8}\|u\|_H^2.$$

Finally, we obtain (5.11) by using the fact that

$$\begin{aligned}\langle q, u\rangle_{L_2} &\leq \|q\|_{L_{4/3}}\|u\|_{L_4} \leq C_4\|q\|_{L_{4/3}}\|u\|_H \\ &\leq C_4^2 a\|q\|_{L_{4/3}}^2 + a^{-1}\|u\|_H^2,\end{aligned}$$

with

$$(5.15)\qquad a = \frac{1}{16}\left(1 + C(M)\mu(A_0)^{-1}\left(1 + \|P_-F_{A_0}\|_{L_3}^3\right)\right).$$

Proof of Lemma 4.6. Set $q \equiv 0$ in (5.10) and (5.11).

Lemma 5.5. *For $q \in L_{4/3}$, the functional $S_q[\cdot]$ is differentiable on $\mathcal{H}$, and*

$$(5.16)\qquad \operatorname{grad} S_q[u; v] \equiv D_v S_q[u] = \left\langle \mathfrak{D}_A^\natural v, \mathfrak{D}_A^\natural u\right\rangle_{L_2} - \langle v, q\rangle_{L_2}$$

is jointly continuous in v, $u \in \mathcal{H}$.

Proof. The difference quotient for the directional derivative for smooth u, v is

$$t^{-1}\left(S_q[u + tv] - S_q[u]\right) = D_v S_q[u] + tS_0[v].$$

Hence (5.16) is valid for $u, v \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$. The extension of (5.16) to $u, v \in \mathcal{H}$ is straightforward and is similar to the proof of Lemma 5.3.

Lemma 5.6. *Assume that the conditions of Proposition* 5.1 *are met. Then the functional $S_q[u]$ is strictly convex.*

Proof. Since $u \to \langle u, q\rangle_{L_2}$ is linear and continuous, it is convex. We prove that $u \to \|\mathfrak{D}_{A_0}^\natural u\|_{L_2}^2$ is strictly convex. The quadratic functional $\|\mathfrak{D}_{A_0}^\natural u\|_{L_2}^2$ is necessarily convex (cf. [16, §VI, 7.9]). By Lemma 4.6, it is an equivalent norm on $\mathcal{H}$, so it is strictly convex.

Proof of Proposition 5.1 (*completion*). The functional $S_q[\cdot]$ is differentiable strictly convex and hence weakly lower semicontinuous. It satisfies the bound (5.11) so by standard arguments [16, §VI, 8.5] it has a unique critical point $u \in \mathcal{H}$, and u minimizes $S_q[\cdot]$ on $\mathcal{H}$. Thus (5.5) holds. For $q \in C^\infty$, standard arguments give $u \in C^\infty$, [18].

The proof of Theorem 4.1 is completed when the apriori estimates (4.3)–(4.5) are established.

Proposition 5.7. *Let $A_0 \in \mathcal{C}(P)$ and suppose that the conditions of Proposition 5.1 are satisfied with $q \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$. Let $u \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$ be the unique solution to* (5.1). *Then*

$$\|\mathcal{D}^\natural_{A_0} u\|_{L_2} \leqslant C_1 \zeta(A_0)^{-1} \|q\|_{L_{4/3}}, \tag{5.17a}$$

$$\|\mathcal{D}^\natural_{A_0} u\|_{H} \leqslant C_1 \left\{ \|q\|_{L_2} + \zeta(A_0)^{-1} \|q\|_{L_{4/3}} \left(1 + \|F_{A_0}\|_{L_4}\right) \right\}, \tag{5.17b}$$

$$\|\mathcal{D}^\natural_{A_0} u\|_{L_4} \leqslant C_1 \left\{ \|q\|_{L_2} + \zeta(A_0)^{-1} \|q\|_{L_{4/3}} \left(1 + \|F_{A_0}\|_{L_4}\right) \right\}, \tag{5.17c}$$

where $C_1 < \infty$ is independent of $A_0 \in \mathcal{C}(P)$, P and q.

Remark. If it is known only that $q \in L_{4/3} \cap L_2$, then (5.17a)–(5.17c) are true for the unique weak solution $u \in \mathcal{H}$ to (5.1). Further, $\mathcal{D}^\natural_{A_0} u \in \mathcal{K}$. Since this generality is not required for the proof of Theorem 4.1, Proposition 5.7 will be proved with the stated assumption that $q \in \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2)$.

Proof of Proposition 5.7. Note first that (5.17c) follows from (5.17b) by using Lemma 5.2. To prove (5.17a), use (5.5) with $v = u$ to obtain

$$\|\mathcal{D}^\natural_{A_0} u\|^2_{L_2} = \langle u, q\rangle_{L_2} \leqslant \|u\|_{L_4} \|q\|_{L_{4/3}}. \tag{5.18}$$

The last step uses Hölder's inequality. Now use Lemmas 5.2 and 4.6. As for (5.17b). Let $b = \mathcal{D}^\natural_{A_0} u$. Then b satisfies

$$\mathcal{D}_{A_0} b = q \tag{5.19a}$$

$$\nabla^\natural_{A_0} b = -*\left(P_-F_{A_0} \wedge u - u \wedge P_-F_{A_0}\right), \tag{5.19b}$$

where $\nabla^\natural_{A_0}\colon \Gamma(\hat{\mathfrak{g}} \otimes \Lambda_1) \to \Gamma(\hat{\mathfrak{g}})$ is the adjoint of $\nabla_{A_0}\colon \Gamma(\hat{\mathfrak{g}}) \to \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$.

The estimate of $\|b\|_H$ comes from the integrated form of (2.9). Substitute $b = \mathcal{D}^\natural_{A_0} u$ in (2.9), and take the L_2 inner product of both sides with b. Integrating by parts and using (5.19a), (5.19b) with Hölder's inequality yields

$$\begin{aligned} \|\nabla_{A_0} b\|^2_{L_2} &\leqslant 2\|q\|^2_{L_2} + 8\|P_-F_{A_0}\|^2_{L_4} \|u\|^2_{L_4} \\ &\quad + 8\|P_+ F_{A_0}\|_{L_4} \|b\|_{L_4} \|b\|_{L_2} + 2\|\mathfrak{R}\|_{L_\infty} \|b\|^2_{L_2}. \end{aligned} \tag{5.20}$$

Now use (5.17a), Lemma 4.6 and Lemma 5.2 to obain (5.17b). This completes the proof of Theorem 4.1.

6. Moduli spaces

The purpose of this section is to establish results which are necessary for the proof of Theorem 1.3. In addition, Theorem 1.4 is proved. We remarked earlier that Theorems 1.3 and 1.4 are proved by Atiyah, Hitchin and Singer [3] in the cases where the Riemannian curvature of M satisfies $\mathcal{W}_- = 0$ and $s > 0$.

As explained by Atiyah et. al., if a self-dual connection A exists in $\mathcal{C}(P)$, then a one-parameter family of self-dual connections on P defines an element in the first cohomology group $H^1_A(\hat{\mathfrak{g}})$ of the following elliptic complex:

(6.1) $$0 \to \Gamma(\hat{\mathfrak{g}}) \overset{D_A}{\to} \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1) \overset{\mathfrak{D}_A}{\to} \Gamma(\hat{\mathfrak{g}} \otimes P_-\Lambda^2) \to 0.$$

Notice that $\mathfrak{D}_A D_A = [P_-F_A, \cdot] = 0$ because A is self-dual. The aim is to compute $H^1_A(\hat{\mathfrak{g}})$ and then to show that it is the tangent space at A to a local space of moduli. One then shows that the local space of moduli is a Hausdorff manifold.

Theorem 6.1. *Let M and P be as in Theorem* 3.2. *Suppose that $A \in \mathcal{C}(P)$ is self-dual and irreducible, and that $\mu(A) > 0$. Then the orbit of A under* Aut P *is a point in a local moduli space of irreducible self-dual connections. In addition, the moduli space in a neighborhood of A is a Hausdorff manifold of dimension*

(6.2) $$P_1(\hat{\mathfrak{g}}) - \frac{\dim G}{2}(\chi - \tau).$$

If every irreducible self-dual connection in $\mathcal{C}(P)$ satisfies $\mu(\cdot) > 0$, then the space of moduli of irreducible self-dual connections is a global Hausdorff manifold of dimension given by (6.2).

Proof of Theorem 1.4 *assuming Theorem* 6.1. The functional $\mathcal{YM}(\cdot)$ and the condition of self-duality are invariant under pointwise conformal transformations of the Riemannian structure, hence we can assume that $s\delta^{ij} - 3\mathcal{W}_-^{ij}$ is a strictly positive matrix. Under this assumption, Proposition 2.2 and equation (2.8) ensure that $\mathfrak{D}_A \mathfrak{D}_A^\natural$ has strictly positive eigenvalues whenever $\|P_-F_A\|_{L_2}$ is sufficiently small. Thus every self-dual connection in $\mathcal{C}(P)$ satisfies $\mu(\cdot) > 0$.

Proof of Theorem 6.1. The proof is sufficiently similar to that of the case treated by Atiyah, Hitchin and Singer, so we only outline the argument and refer the reader to [3].

The first step is to compute $h^1 = \dim H^1_A(\hat{\mathfrak{g}})$. Since A is irreducible, $h^0 = \ker D_A = 0$. Further $h^2 = 0$ because $h^2 = \ker \mathfrak{D}_A^\natural$;

(6.3) $$\|\mathfrak{D}_A^\natural u\|^2_{L_2} \geq \mu(A)\|u\|^2_{L_2},$$

for all $u \in \mathcal{K}$. We now compute $h^0 - h^1 + h^2$ by the Atiyah-Singer index theorem [4]. The result as in [3] is

(6.4) $$h^1 = p_1(\hat{\mathfrak{g}}) - \frac{\dim G}{2}(\chi - \tau).$$

Since $\mu(A) > 0$, $\mu(\cdot) > 0$ in a neighborhood of $A \in \mathcal{C}(P)$. (This is still true if we give $\mathcal{C}(P)$ the Banach space structure of an L_p^k space for $p \geq 2$, $k \geq 1$.) This remark and (6.3) allow us to conclude that there exists in a neighborhood of A, a local moduli space which is a Hausdorff manifold of dimension h^1. As in [3] one also shows that if $\mu(A) > 0$ for every irreducible self-dual $A \in \mathcal{C}(P)$, then these local moduli spaces give local coordinates on a global moduli space, and that this global space is a Hausdorff manifold.

7. Self-dual connections: S^4 and $\mathbf{R}^4$

We have yet to produce connections which satisfy the conditions of Theorem 3.2. In the next section, these connections will be explicitly constructed; the result is Theorem 8.2. This construction is a "cut and paste" operation which uses the self-dual connections on S^4. For this reason it is helpful to review their properties. Their existence has previously been established: Theorem 7.1 lists the principal bundles which admit self-dual connections. The self-dual connections over S^4 pull back via stereographic projection to self-dual connections on $\mathbf{R}^4$. Theorem 7.4 and Corollary 7.5 establish apriori estimates on the size of their curvature as $|x| \to \infty$ on $\mathbf{R}^4$. These estimates are crucial to the patching theorems in the next section. Finally, Proposition 7.7 summarizes the behavior of self-dual connections under scale transformations on $\mathbf{R}^4$.

When G is a compact semi-simple Lie group, the question of classifying all irreducible self-dual G-connections over S^4 has been solved. It was pointed out by Atiyah, Hitchin and Singer [3] that it is only necessary to consider groups G which are simple and simply connected. The reason is the following: If $P \to S^4$ is a principal G-bundle, then a connection on P has a unique lifting to a connection on the universal covering group bundle (see the Appendix). The universal covering group bundle is a direct product of principal bundles with structure groups which are simple and simply connected Lie groups. This means that the connection on the universal covering group bundle is a direct sum of connections on the bundles which make up this direct product. By Proposition A.1, a principal bundle over S^4 with simple and simply connected structure group is classified by an integer k, called the Pontrjagin index ($k = p_1(\hat{\mathfrak{g}}) \cdot r_{\mathfrak{g}}^{-1}$ with $r_{\mathfrak{g}}$ given in (A.5)).

Theorem 7.1. (*Atiyah, Hitchin and Singer* [3]). *There exist irreducible self-dual G-connections on S^4 when the associated vector bundle $\hat{\mathfrak{g}}$ has Pontrjagin index k if and only if for $Sp(n)$, $k \geqslant n$; $SU(n)$, $k \geqslant n/2$; $Spin(n)$, $k \geqslant n/4$; G_2, $k \geqslant 2$; F_4, E_6, E_7, E_8, $k \geqslant 3$.*

This theorem, the preceding discussion and Theorem 1.4 solve the existence question on S^4. The following are two useful extensions of Theorem 7.1.

Corollary 7.2. *There exist self-dual connections on a principal G bundle $P \to S^4$ with structure group G compact and simple when the associated vector bundle $\hat{\mathfrak{g}}$ has positive Pontrjagin class.*

Proof. For G compact and simple, there is an embedding $SU(2) \subset G$ which induces an isomorphism of homotopy groups $\Pi_3(SU(2)) \to \Pi_3(G)$, [7]. Hence G-bundle of index k over S^4 are reducible to $SU(2)$-bundles with the same index. Therefore self-dual $SU(2)$ connections exist in $\mathcal{C}(P)$ as reducible elements.

Proposition 7.3. *A self-dual connection over S^4 is equivalent via a G-bundle isomorphism to a real analytic connection on a real analytic principal bundle.*

Proof. See, for example [3], [25].

The self-dual connections on S^4 can be pulled back to $\mathbf{R}^4$ to give self-dual connections there. This is our next topic.

Let p denote the north pole of S^4, and $\bar{p}$ the south pole. The open sets $\{U_1 = S^4 \setminus \bar{p}, U_2 = S^4 \setminus p\}$ are a trivializing cover for any bundle $P \to S^4$. Thus if P is a principal G-bundle over S^4, it is uniquely determined by its transition function $h\colon U_1 \cap U_2 \to G$. The connection $A \in \mathcal{C}(P)$ is equivalent to a pair of Lie algebra valued one-forms $A^i \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1|_{U_i})$ which satisfy in $U_1 \cup U_2$

$$A^2 = h^{-1}dh + \mathrm{Ad}(h^{-1})(A^1). \tag{7.2}$$

The curvature of A is

$$F_A^i = dA^i + A^i \wedge A^i \quad (i = 1,2) \tag{7.3}$$

in U_i, and

$$F_A^2 = \mathrm{Ad}(h^{-1})(F_A{}^1). \tag{7.4}$$

in $U_1 \cap U_2$.

Let $s\colon \mathbf{R}^4 \to U_1$ be the stereographic projection from $\bar{p}$. The map s is a conformal diffeomorphism. Thus if A is a smooth solution to (1.2a) and (1.2b) on S^4, then $s^*(A)$ is a smooth solution to the same equations on $\mathbf{R}^4$. If we denote the canonical flat G-connection on $\mathbf{R}^4$ by Γ_0, then

$$s^*(A) = \Gamma_0 + s^*(A^1), \tag{7.5a}$$

$$F_{s^*(A)} = s^*(F_A{}^1). \tag{7.5b}$$

Theorem 7.4 (*Uhlenbeck* [25]). *Let G be compact and semi-simple. Let A be a smooth connection on a principal G-bundle over S^4 which satisfies* (1.2*a*) *and* (1.2*b*). *Then*

$$\|s^*(F_A)\|_{L_2;\mathbf{R}^4} \leqslant \infty,$$

$$|s^*(F_A)|(x) \leqslant \frac{K}{(1+|x|^2)^2}, \tag{7.6}$$

where K is a finite constant which depends on A.

Corollary 7.5. *Let A be as in Theorem 7.4. There exists a gauge transformation $g \in \mathcal{G}(S^4 \setminus \{p, \bar{p}\}; G)$ such that*

$$|s^*(g(A)) - \Gamma_0| \leq \frac{1}{2}K\frac{1}{|x|\left(1+|x|^2\right)} \tag{7.7}$$

for $x \in \mathbf{R}^4 \setminus \{0\}$.

Proof. Let $\bar{s}: \mathbf{R}^4 \to U_2$ be the stereographic projection from p. By making a smooth gauge transformation $g_2 \in \Gamma(U_2; G)$ we arrange that

$$\overline{A}^2 = g_2^{-1}dg_2 + \mathrm{Ad}\left(g_2^{-1}\right)(A^2) \tag{7.8}$$

satisfies

$$\overline{A}^2(\bar{p}) = 0, \quad s_*(\hat{x}) \lrcorner \overline{A}^2 = 0, \tag{7.9}$$

where $\hat{x} = x^\nu(\partial/\partial x^\nu) \in \Gamma(T^*|_{\mathbf{R}^4})$. K. Uhlenbeck proved that g_2 always exists.

Now let

$$g = g_2 h. \tag{7.10}$$

Then

$$s^*(g(A)) = \Gamma_0 + s^*(\overline{A}^2). \tag{7.11}$$

In order to estimate $|s^*(\overline{A}^2)|$ we use the fact [25] that

$$(\bar{s}^*(\overline{A}^2))(x) = \int_0^1 d\tau\left(\bar{s}^*\left(\bar{s}_*(\hat{x}) \lrcorner F_A{}^2\right)\right)(\tau x). \tag{7.12}$$

(7.7) follows from (7.12), in consequence of (7.6) and the relation

$$\bar{s}^*((s^{-1})^*(x^\nu)) = \frac{x^\nu}{|x|^2}, \tag{7.13}$$

valid for $x \in \mathbf{R}^4 \setminus \{0\}$.

Definition 7.6. For $\lambda > 0$, the scale transformation $\lambda: \mathbf{R}^4 \to \mathbf{R}^4$ is given by

$$\lambda^*(x^\nu) = x^\nu/\lambda. \tag{7.14}$$

The Yang-Mills functional (1.1) on $\mathbf{R}^4$ is scale invariant. Thus if A is a solution to (1.2a) and (1.2b) on $\mathbf{R}^4$, then $\lambda^*(A)$ is also a solution, and

$$\mathcal{YM}(\lambda^*(A)) = \mathcal{YM}(A). \tag{7.15}$$

We see that λ^* maps self-dual connections into self-dual connections. However, λ^* affects the C^0 norm of a connection in the following way.

Proposition 7.7. *Let $\lambda \in (0, \infty)$. Let A and g be as in Theorem 7.4 and Corollary 7.5 respectively. Then the following are true:*

(7.16)

(a) $|\lambda^*(s^*(F_A))|(x) \leq \dfrac{\lambda^2 K}{\left(\lambda^2+|x|^2\right)^2}$.

(b) If $x \in \mathbf{R}^4 \setminus \{0\}$, *then*

$$|\lambda^*(s^*(g(A))) - \Gamma_0|(x) \leq \frac{1}{2}\frac{\lambda^2 K}{|x|(\lambda^2 + |x|^2)}. \tag{7.17}$$

Proof. Use the fact that if ω is a one-form on $\mathbf{R}^4$, then

$$(\lambda^*(\omega))(x) = \lambda^{-1}\omega(\lambda^{-1}x).$$

8. Almost self-dual connections

In this section, M is any compact orientable Riemannian manifold, and G is a compact semi-simple Lie group. A principal G-bundle $P \to M$ and a connection $A \in \mathcal{C}(P)$ satisfying the requirements of Theorem 3.2 will be constructed. An outline of this construction follows: Choose a point $m \in M$ and Gaussian normal coordinate system centered at m. This coordinate system covers a ball B of radius $R > 0$ also centered at m. The coordinate functions provide a diffeomorphism of B to the ball $\bar{B}$ of radius R centered at $\{0\} \in \mathbf{R}^4$. Let s^*W be a self-dual irreducible connection on $\mathbf{R}^4 \times G$, which is the pull back from S^4 of a self-dual connection W on the principal G-bundle over S^4 with the requisite Pontrjagin classes. The pull back is via stereographic projection. By using the scale transformation (7.14), we can demand that $\lambda^*s^*(W)$ have most of its curvature in $\bar{B}$. We then modify $\lambda^*s^*(W)$ to produce a connection $\tilde{W}$ which is flat outside $\bar{B}$. Finally we define A to be the pull back of $\tilde{W}$ in B, and to be flat in $M \setminus B$. This serves to define the bundle P as well. The associated vector bundle $\hat{\mathfrak{g}}$ has the correct Pontrjagin classes, and by adjusting λ one can make $\delta(A)$ of Definition 3.1 arbitrarily small.

The condition $\delta(A) < \delta$ is an open condition on $\mathcal{C}(P)$, so apriori, there are irreducible connections $A \in \mathcal{C}(P)$ with $\delta(A) < \delta$. A measure of irreducibility is required. Because M is compact, there exists $\rho_0 > 0$ such that for each $m \in M$, the open ball $B_\rho(M)$ of radius $\rho_0 > \rho > 0$ centered at m is diffeomorphic to the unit ball in $\mathbf{R}^4$. For $\rho_0 \geq \rho > 0$ and $m \in M$ let

$$T(\rho, m) = \left\{\sigma \in \Gamma\big(\bar{B}_\rho(m); \hat{\mathfrak{g}}\big) : \int_{B_\rho(m)} *|\sigma|^2 = 1\right\}. \tag{8.1}$$

Definition 8.1. For $A \in \mathcal{C}(P)$ and $m \in M$, define

$$\mathcal{J}_m(A) = \sup_{\rho_0 \geq \rho > 0}\left(\inf_{\sigma \in T(\rho, m)}\left(\int_{B_\rho(m)} *|\nabla_A\sigma|^2\right)\right). \tag{8.2}$$

Clearly, $\mathcal{J}_m(A)$ is finite, for if $\sigma_0 \in \Gamma(B_{\rho_0}(m); \hat{\mathfrak{g}})$ satisfies $|\sigma_0|(x) = 1$, then

$$\mathcal{J}_m(A) \leq \|\nabla_A\sigma_0\|^2_{L_\infty}. \tag{8.3}$$

On the other hand,

$$(8.4)\qquad \mathcal{J}_m(A) \geqslant \inf_{\sigma \in T(\rho_0, m)} \left(\int_{B_{\rho_0}(m)} * |\nabla_A \sigma|^2 \right).$$

Standard techniques from the calculus of variations [18] allow us to conclude that the infimum on the left-hand side of (8.4) is achieved by some $\omega \in T(\rho_0, m)$. What is important is that A is reducible only if $\mathcal{J}_m(A) = 0$ for all $m \in M$.

The measure $\mathcal{J}_m(A)$ is used in the following theorem.

Theorem 8.2. *Let M be a compact oriented Riemannian manifold, and let G be a compact semi-simple Lie group. Suppose that $p \to M$ is a principal G-bundle, all of whose Pontrjagin classes are nonnegative. In addition, suppose that the image of the isomorphism class of P in $H^2(M; \Pi_1(G))$ is trivial* (*see the Appendix.*) *Given $\delta > 0$.*

(i) *There exists $A \in \mathcal{C}(P)$ with $\|P_-F_A\|_{L_p} < z_1\delta^{1/p}$, with z_1 independent of δ*

(ii) *Suppose that the principal G-bundle $P' \to S^4$ with the same Pontrjagin classes as P admits an irreducible self-dual connection. Then there is a constant $z > 0$ which is independent of δ with the following significance: There exists $A \in \mathcal{C}(P)$ such that $\delta(A) < \delta$ and for some $m \in M$, $\mathcal{J}_m(A) > z$.*

(iii) *If $P_-H^2_{\text{De Rham}}(M) = 0$, there exists a constant $\alpha > 0$ which is independent of δ with the following properties: There exist $A \in \mathcal{C}(P)$ with $\delta(A) < \delta$, $\mu(A) > \alpha$, and in addition A satisfies* (i) *and* (ii).

The reader is referred back to §7 and Theorem 7.1 for the conditions where (ii) is applicable. Theorem 8.2 supercedes Theorem 1.6.

The proof of Theorem 8.2 requires the introduction of a function $\beta \in C^\infty(\mathbf{R}^4)$ with the following properties:

$$(8.5)\qquad \begin{aligned} &0 \leqslant \beta(x) \leqslant 1, \\ &\beta(x) = 1 \qquad \text{if } |x| < 1, \\ &\beta(x) = 0 \qquad \text{if } |x| > 3/2. \end{aligned}$$

For $r > 0$, define $\beta_r(x) = \beta(x/r)$.

Proof of Theorem 8.2. A principal bundle is uniquely determined by its transition functions, so P will be defined by giving an open cover $\{V_b\}_{b\in\Lambda}$ of M and functions $\{g_{b,b'}\colon V_b \cap V_{b'} \to G\}_{b,b'\in\Lambda}$, where Λ is a finite indexing set. The functions $\{g_{b,b'}\}$ satisfy $g_{b,b} = 1_G$ as well as the cocycle condition

$$(8.6)\qquad g_{b,b'}g_{b'b''}g_{b'',b} = 1_G$$

in $V_b \cap V_{b'} \cap V_{b''}$.

The connection $A \in \mathcal{C}(P)$ will be defined by a set $\{A^b \in \Gamma(V_b; \hat{\mathfrak{g}} \otimes \Lambda^1)\}_{b \in \Lambda'}$ which respects the cocycle condition

$$A^{b'} = g_{b,b'}^{-1} dg_{b,b'} + \mathrm{Ad}\left(g_{b,b'}^{-1}\right)\left(A^b\right) \tag{8.7}$$

in $V_b \cap V_{b'}$.

Fix a point $m \in M$. There exists a coordinate neighborhood $U \ni m$ and a coordinate chart

$$\phi\colon U \to \mathbf{R}^4 \tag{8.8}$$

with the following properties

(i) $\phi(m) = \{0\} \in \mathbf{R}^4$;

(ii) the components of the Riemannian metric, as defined by

$$g^{\mu\nu}(m') = \left(\phi^*(dx^\mu), \phi^*(dx^\nu)\right)(m'), \tag{8.9}$$

satisfy

$$\begin{gathered} g^{\mu\nu}(m) = \delta^{\mu\nu}, \\ (dg^{\mu\nu})(m) = 0, \\ |g^{\mu\nu}(m') - \delta^{\mu\nu}| \leqslant |\phi(m')|^2 \rho(m) \end{gathered} \tag{8.10}$$

for all $m' \in U$, where $\rho(m)$ is a finite constant which depends on the Riemannian curvature of M [17]. Choose $R > 0$ and sufficiently small so that

$$\zeta \equiv R^2 \rho(m) \ll 1, \tag{8.11}$$

and set $B_R = \{m' \in U\colon |\phi(m')| < R\}$. Then for all $m' \in B_R$,

$$|g^{\mu\nu}(m') - \delta^{\mu\nu}| \leqslant \zeta \ll 1. \tag{8.12}$$

Let $P' \to S^4$ be a principal G-bundle such that the Pontrjagin classes of the associated vector bundle $\hat{\mathfrak{g}}'$ are all nonnegative. By Theorem 7.1 and Corollary 7.2, there exists a self-dual connection $W \in \mathcal{C}(P')$. Using the notation of §7, W defines one-forms $\{W^i \in \Gamma(U_i; G)\}_{i=1}^2$ where in $U_1 \cap U_2$,

$$W^2 = h^{-1} dh + \mathrm{Ad}(h^{-1})(W^1), \tag{8.13}$$

and $h \in \Gamma(U_1 \cap U_2\colon G)$ is the transition function. The connection W defines the gauge transformation $g \in \Gamma(U_1 \cap U_2; G)$, which is given in Corollary 7.5, and the one-form $\overline{W}^2 \in \Gamma(U_2; \hat{\mathfrak{g}} \otimes \Lambda^1)$:

$$\overline{W}^2 = g^{-1} dg + \mathrm{Ad}(g^{-1})(W^1). \tag{8.14}$$

Let $\lambda\colon \mathbf{R}^4 \to \mathbf{R}^4$ denote the scale transformation of Definition 7.6, and let $s\colon \mathbf{R}^4 \to U_1$ be stereographic projection.

Definition 8.3. *The bundles* P_λ: These are defined for $\lambda \in (0, \min(1, \frac{1}{4}R^2))$. Cover M by the two open sets

$$\{V_1 = B_{\sqrt{\lambda}}(m), V_2 = M \setminus m\}. \tag{8.15}$$

The transition function in $V_1 \cap V_2$ is

$$g_{1,2} = \phi^*(\lambda^*(s^*g)). \tag{8.16}$$

Definition 8.4. *The connections* A_λ: These are also defined for $\lambda \in (0, \min(1, \frac{1}{4}R^2))$ as connections on P_λ. In V_1, set

$$A^1 = \phi^*\big(\lambda^*(s^*(W^1))\big), \tag{8.17}$$

and in V_2 set

$$A^2 = \phi^*\big(\beta_{\sqrt{\lambda}} \cdot \lambda^*(s^*(\overline{W}^2))\big). \tag{8.18}$$

Notice that (8.14) and (8.16) ensure that $(A^1_\lambda, A^2_\lambda)$ satisfy the cocycle condition (8.7) in $V_1 \cap V_2$.

The first properties of (P_λ, A_λ) to calculate are the Pontrjagin classes.

Proposition 8.5. *The vector bundle $\hat{\mathfrak{g}}_\lambda$ which is associated to P_λ via* Ad_G *has the same Pontrjagin classes as the vector bundle $\hat{\mathfrak{g}}' \to S^4$ which is associated to P' via* Ad_G. *Further, the image of the isomorphism class of P_λ in $H^2(M; \Pi_1(G))$ under the map η is the trivial element.*

Proof. Both statements follow from the functorial properties of the set {isomorphism classes of principal G-bundles over M} and the fact that $P_\lambda \to M$ is the pull back of $P' \to S^4$ via a degree 1 map from M onto S^4.

As for $P\_F_{A_\lambda}$, one has the following upper bound:

Proposition 8.6. *There exists a constant $z_1 < \infty$ which is independent of λ such that for $p \in [1, \infty)$,*

$$\begin{aligned} \|P_- F_{A_\lambda}\|_{L_p} &\leq z_1 \lambda^{2/p}, \\ \|F_{A_\lambda}\|_{L_p} &\leq z_1 \lambda^{4/p-2}. \end{aligned} \tag{8.19}$$

Proof. We are required to estimate $\|F_{A_\lambda}\|_{L_p}$, $\|P\_F_{A_\lambda}\|_{L_p}$. This is done by breaking M into the three sets $M \setminus B_{2\sqrt{\lambda}}$, $B_{2\sqrt{\lambda}} \setminus B_{\sqrt{\lambda}}$ and $B_{\sqrt{\lambda}}$, and computing the integrals over each set separately. In fact, since $F_A = 0$ in $M \setminus B_{2\sqrt{\lambda}}$, only $B_{2\sqrt{\lambda}} \setminus B_{\sqrt{\lambda}}$ and $B_{\sqrt{\lambda}}$ need be considered. The set $B_{2\sqrt{\lambda}}$ is diffeomorphic via the coordinate chart ϕ to the ball of radius $2\sqrt{\lambda}$ in $\mathbf{R}^4$. Therefore the metric tensor $g^{\mu\nu}$ can be pulled back using ϕ^{-1} to $\phi(B_{2\sqrt{\lambda}})$, and the calculation can be done there. Let $|\cdot|_g$ denote the pointwise inner product which is defined by this pulled back metric (which we still denote by $g^{\mu\nu}$). The norm defined by the flat Euclidean metric is denoted $|\cdot|$. To avoid confusion '$*g$' will denote the Hodge duality operator which is defined by $g^{\mu\nu}$. For notational convenience, we denote $y \in B_{2\sqrt{\lambda}}$ and $x = \phi(y)$ by x.

Fix $x \in B_{\sqrt{\lambda}}$. Using (8.10) and (8.11) we conclude that

$$|F_{A_\lambda} - *gF_{A_\lambda}|_g(x) \leq k_1 |x|^2 |F_{A_\lambda}|(x), \tag{8.20}$$

$$|F_{A_\lambda}|_g(x) \leqslant k_1 |F_{A_\lambda}|(x), \tag{8.21}$$

where k_1 is a finite constant which is independent of λ. In fact, for $x \in B_{\sqrt{\lambda}}$,

$$F_{A_\lambda} = \phi^*(\lambda^*(s^*(F_W))) \tag{8.22}$$

is self-dual with respect to the flat metric. We now use Proposition 7.7 to obtain the following estimates:

$$|F_A - {}^*g\, F_{A_\lambda}|_g(x) \leqslant k_1 K |x|^2 \frac{\lambda^2}{\left(\lambda^2 + |x|^2\right)^2} \leqslant \tfrac{1}{4} k_1 K, \tag{8.23}$$

$$|F_{A_\lambda}|_g(x) \leqslant k_1 K \frac{\lambda^2}{(x^2+\lambda^2)^2}, \; x \in B_{\sqrt{\lambda}}. \tag{8.24}$$

Inequalities (8.23) and (8.24) along with (8.10) and (8.11) imply the integral bounds as follows:

$$\left(\int_{|x|<\sqrt{\lambda}} \sqrt{g}\, d^4 |F_{A_\lambda} - {}^*g\, F_{A_\lambda}|_g^n(x)\right)^{1/n} \leqslant k_2 \lambda^{2/n}, \tag{8.25}$$

$$\left(\int_{|x|<\sqrt{\lambda}} \sqrt{g}\, d^4x\, |F_{A_\lambda}|_g^n(x)\right)^{1/n} \leqslant k_2 \lambda^{4/n-2}, \tag{8.26}$$

where $\sqrt{g} = (\det g^{\mu\nu})^{1/2}$, and k_2 is a finite constant which is independent of λ.

Because $B_{2\sqrt{\lambda}} \setminus B_{\sqrt{\lambda}} \subset V_2$, the curvature of A_λ at a point $x \in B_{2\sqrt{\lambda}} \setminus B_{\sqrt{\lambda}}$ can be computed from A_λ^2 which is given by (8.18). Thus we find that in $B_{2\sqrt{\lambda}} \setminus B_{\sqrt{\lambda}}$,

$$\begin{aligned} F_{A_\lambda} = \phi^*\big(&\beta_{\sqrt{\lambda}} \lambda^*(s^*(\bar{F}_W)) + d\beta_{\sqrt{\lambda}} \wedge \lambda^*(s^*(\bar{W}^2)) \\ &- \beta_{\sqrt{\lambda}}(1-\beta_{\sqrt{\lambda}})\lambda^*(s^*(\bar{W}^2 \wedge \bar{W}^2))\big), \end{aligned} \tag{8.27}$$

where $\bar{F}_W = d\bar{W}^2 + \bar{W}^2 \wedge \bar{W}^2$. An upper bound on the norm $|F_{A_\lambda}|_g$ in $B_{2\sqrt{\lambda}} \setminus B_{\sqrt{\lambda}}$ follows from (8.11), Proposition 7.7 and the scaling relation

$$|d\beta_{\sqrt{\lambda}}|(x) = \frac{1}{\sqrt{\lambda}} |d\beta|\left(x/\sqrt{\lambda}\right). \tag{8.28}$$

Since $s^*(\bar{W}^2) = s^*(g(A)) - \Gamma_0$, we obtain

$$\begin{aligned} &|F_{A_\lambda}|_g(x) \\ &\leqslant k_3 \left(\frac{\lambda^2}{(x^2+\lambda^2)^2} + \frac{\lambda^{3/2}}{|x|\left(|x|^2+\lambda^2\right)} + \frac{\lambda^4}{|x|^2\left(\lambda^2+|x|^2\right)^2} \right) \end{aligned} \tag{8.29}$$

for $x \in B_{2\sqrt{\lambda}} \setminus B_{\sqrt{\lambda}}$, where k_3 is a finite constant which is independent of λ. The three terms in (8.29) correspond to the three terms in (8.27). Because $\lambda \leqslant 1$ and

$x > \sqrt{\lambda}$, (8.29) implies that

$$(8.30) \qquad |F_{A_\lambda}|_g \leqslant k_4 \quad \text{if } \sqrt{\lambda} < |x| < 2\sqrt{\lambda}.$$

The finite constant k_4 is independent of λ. Hence we obtain the integral inequality

$$(8.31) \qquad \left(\int_{\sqrt{\lambda}<|x|<2\sqrt{\lambda}} \sqrt{g}\, d^4x\, |F_{A_\lambda}|_g^n\right)^{1/n} < k_5 \lambda^{2/n}.$$

Putting (8.25), (8.26) and (8.31) together yields

$$(8.32) \qquad \| P_- F_{A_\lambda} \|_{L_p} \leqslant z_1 \lambda^{2/p}, \quad \| F_{A_\lambda} \|_{L_p} \leqslant z_1 \lambda^{4/p-2},$$

where z_1 does not depend on λ.

Now assume that $P^1 \to S^4$ admits an irreducible self-dual connection.

Proposition 8.7. *If W of Definition* 8.4 *is irreducible, then*

$$(8.33) \qquad \inf_{\sigma \in T(\lambda, m)} \int_{B_\lambda(m)} * |\nabla_{A_\lambda} \sigma|^2 \geqslant z > 0,$$

and z is independent of λ.

Proof. Because $s^*(W)$ is gauge equivalent to a real analytic connection and is ireducible,

$$(8.34) \qquad \left(\int_{|x|\leqslant 1} d^4x\, |\sigma|^2\right)^{-1} \left(\int_{|x|\leqslant 1} d^4x\, |\nabla_{s^*(W)}\sigma|^2\right) \geqslant z_2 > 0$$

for all nonzero $\sigma: \mathbf{R}^4 \to \mathfrak{g}$. By rescaling the integrand in (8.34), we obtain for $\lambda \in (0,1]$ that

$$(8.35) \quad \lambda^2 \left(\int_{|x|\leqslant\lambda} d^4x\, |\lambda * (\sigma)|^2\right)^{-1} \left(\int_{|x|\leqslant\lambda} d^4x\, |\nabla_{\lambda^*(s^*(W))} \lambda^*(\sigma)|^2\right) \geqslant z_2 > 0.$$

Hence, for all nonzero $\sigma: \mathbf{R}^4 \to \mathfrak{g}$,

$$(8.36) \qquad \left(\int_{|x|\leqslant\lambda} d^4x\, |\sigma|^2\right)^{-1} \left(\int_{|x|\leqslant\lambda} d^4x\, |\nabla_{\lambda^*(s^*(W))}\sigma|^2\right) \geqslant z_2 > 0.$$

Using (8.12) we conclude that for all nonzero $\sigma: \mathbf{R}^4 \to \mathfrak{g}$,

$$(8.37) \quad \left(\int_{|x|\leqslant\lambda} \sqrt{g}\, d^4x\, |\sigma|^2\right)^{-1} \left(\int_{|x|\leqslant\lambda} \sqrt{g}\, d^4x\, |\nabla_{\lambda^*(s^*(W))}\sigma|_g^2\right) \geqslant z > 0,$$

where z is independent of λ. This last expression is just (8.33) which proves Proposition 8.7.

Next assume that $P_- H^2_{\text{De Rham}}(M) = 0$.

Proposition 8.8. *If $P\_H^2_{\mathrm{De\,Rham}}(M) = 0$, there exist constants $\gamma, \alpha > 0$ which are independent of λ such that if $\lambda < \gamma$ then $\mu(A_\lambda) > \alpha$.*

Proof. By construction, A_λ is flat over $M \setminus B_{2\sqrt{\lambda}}$; as bundles with connection,

$$\hat{\mathfrak{g}} \otimes \Lambda^p \Big|_{M/B_{2\sqrt{\lambda}}} \cong \mathfrak{g} \times \Lambda^p \Big|_{M\setminus B_{2\sqrt{\lambda}}}, \qquad D_{A_\lambda}\Big|_{M\setminus B_{2\sqrt{\lambda}}} = d\Big|_{M\setminus B_{2\sqrt{\lambda}}}. \tag{8.38}$$

Define a map $^-: \Gamma(\hat{\mathfrak{g}}_\lambda \otimes \Lambda^p) \to \Gamma(\mathfrak{g} \times \Lambda^p)$ by

$$\psi \to \bar{\psi} \equiv (1 - \beta_{2\sqrt{\lambda}})\psi. \tag{8.39}$$

Then $\psi = \beta_{2\sqrt{\lambda}}\psi + \bar{\psi}$, and for $\psi \in \Gamma(\mathfrak{g} \otimes P\_\Lambda^2)$,

$$\|\mathfrak{D}^\natural_{A_\lambda}\psi\|^2_{L_2} = \|d^\natural\bar{\psi}\|^2_{L_2} + 2\left\langle \mathfrak{D}^\natural_{A_\lambda}\bar{\psi}, \mathfrak{D}^\natural_{A_\lambda}(\beta_{2\sqrt{\lambda}}\psi)\right\rangle_{L_2} + \|\mathfrak{D}^\natural_{A_\lambda}(\beta_{2\sqrt{\lambda}}\psi)\|^2_{L_2}. \tag{8.40}$$

Now suppose that ψ is a L_2-normalized eigenvector of $\mathfrak{D}_{A_\lambda}\mathfrak{D}^\natural_{A_\lambda}$ with eigenvalue μ. We want to derive a lower bound for μ. To estimate the last term in (8.40) we use (5.4), (5.2) and Lemma 5.2. Thus

$$\begin{aligned} &\|\mathfrak{D}^\natural_{A_\lambda}(\beta_{2\sqrt{\lambda}}\psi)\|^2_{L_2} \\ &\geq \frac{1}{4}\|\nabla|\beta_{2\sqrt{\lambda}}\psi|\|^2_{L_2} - C(M)(\lambda + \|P\_F_{A_\lambda}\|_{L_2})\|\beta_{2\sqrt{\lambda}}\psi\|^2_{L_4}, \end{aligned} \tag{8.41}$$

where $C(M)$ is a constant depending on the Riemannian structure of M, and we have used the fact that the support $(\beta_{2\sqrt{\lambda}}\psi) \subset B_{3\sqrt{\lambda}}(m)$. Standard Sobolev inequalities [14] imply that

$$\|\nabla|\beta_{2\sqrt{\lambda}}\psi|\|^2_{L_2} \geq C_2(M)\lambda^{-1}\|\beta_{2\sqrt{\lambda}}\psi\|^2_{L_2} + C_4(M)\|\beta_{2\sqrt{\lambda}}\psi\|^2_{L_4}. \tag{8.42}$$

Hence using (8.42) and Proposition (8.6) we obtain, for λ sufficiently small, the bound

$$\|\mathfrak{D}^\natural_{A_\lambda}(\beta_{2\sqrt{\lambda}}\psi)\|^2_{L_2} \geq C_2(M)\lambda^{-1}\|\beta_{2\sqrt{\lambda}}\psi\|^2_{L_2}. \tag{8.43}$$

As for the second term in (8.40), we use the identities

$$\begin{aligned} \mathfrak{D}^\natural_{A_\lambda}(\beta_{2\sqrt{\lambda}}\psi) &= *\,(d\beta_{2\sqrt{\lambda}} \wedge \psi) + \beta_{2\sqrt{\lambda}}\mathfrak{D}^\natural_{A_\lambda}\psi, \\ \|d\beta_{2\sqrt{\lambda}} \wedge \psi\|_{L_2} &\leq C_3\lambda^{1/2}\|\psi\|_{L_\infty}, \end{aligned} \tag{8.44}$$

to obtain the bound

$$\langle \mathfrak{D}^\natural_{A_\lambda}\bar{\psi}, \mathfrak{D}^\natural_{A}(\beta_{2\sqrt{\lambda}}\psi)\rangle_{L_2} \leq \|\mathfrak{D}^\natural_{A_\lambda}\bar{\psi}\|_{L_2}\left(\|\mathfrak{D}^\natural_{A_\lambda}\psi\|_{L_2} + C_3\lambda^{1/2}\|\psi\|_{L_\infty}\right). \tag{8.45}$$

To estimate $\|\Psi\|_{L_\infty}$ we must use the eigenvalue equation in integrated form, namely for all $\eta \in \mathcal{H}$

$$\langle \mathfrak{D}_A^\natural \eta, \mathfrak{D}_A^\natural \psi \rangle - \mu \langle \eta, \psi \rangle = 0,$$

or

$$(8.46) \qquad \begin{aligned} &\langle \nabla_{A_\lambda}\eta, \nabla_{A_\lambda}\psi\rangle_{L_2} + \sqrt{2}\langle \eta, P_-F_{A_\lambda}(\psi)\rangle_{L_2} + \frac{2}{3}\langle \eta, s\psi\rangle_{L_2} \\ &\quad -2\langle \eta, \mathcal{W}_-(\psi)\rangle_{L_2} - \mu\langle \eta, \psi\rangle_{L_2} = 0. \end{aligned}$$

Let $0 \leq f \in C^\infty(M; \mathbf{R})$, and set $v = (1 + |\psi|^2)^{1/2}$ and $\eta = fv^{-1}\psi$. Since

$$\nabla_A \eta = \nabla f v^{-1}\psi - fv^{-2}\nabla v \psi + fv^{-1}\nabla\psi,$$

we obtain that

$$(8.47) \qquad \langle \nabla f, \nabla v\rangle_{L_2} - \sqrt{2}\langle vf | P_-F_{A_\lambda}|\rangle_{L_2} - (c_3 + \mu)\langle v, f\rangle_{L_2} \leq 0,$$

where c_3 depends only on the Riemannian structure of M. Appealing to Morrey [18, Theorem 5.3.1], we obtain a uniform bound on $\|v\|_{L_\infty}$ of the form

$$(8.48) \qquad \|\psi\|_{L_\infty} \leq \|v\|_{L_\infty} \leq c_4(M; \mu)\big(\|\psi\|_{L_2} + 1\big) \leq 2c_4\|\psi\|_{L_2},$$

since we have normalized ψ so that $\|\psi\|_{L_2} = 1$. Hence the right-hand side of (8.45) is bounded by

$$(8.49) \quad \langle \mathfrak{D}_{A_\lambda}^\natural \bar{\psi}, \mathfrak{D}_{A_\lambda}^\natural(\beta_{2\sqrt{\lambda}}\psi)\rangle_{L_2} \leq C(M)\|\mathfrak{D}_{A_\lambda}^\natural\bar{\psi}\|_{L_2}\big(\|\mathfrak{D}_{A_\lambda}^\natural\psi\|_{L_2} + \lambda^{1/2}\|\psi\|_{L_2}\big),$$

where $C(M)$ is a (different) constant which is independent of ψ and λ.

On the other hand, because $P_-H^2_{\text{De Rham}}(M) = 0$, there exists a constant $\mu_1 > 0$ which is independent of λ such that

$$(8.50) \qquad \|\mathfrak{D}_{A_\lambda}^\natural\bar{\psi}\|_{L_2}^2 = \|d^\natural\bar{\psi}\|_{L_2}^2 \geq \mu_1\|\bar{\psi}\|_{L_2}^2.$$

Together, (8.40), (8.43), (8.47) and (8.48) imply that for λ sufficiently small,

$$(8.51) \quad \|\mathfrak{D}_{A_\lambda}^\natural\psi\|_{L_2}^2 \geq \mu_2\big(\|\bar{\psi}\|_{L_2}^2 + \lambda^{-1}\|\beta_{2\sqrt{\lambda}}\psi\|_{L_2}^2 - \lambda\|\psi\|_{L_2}^2\big) \geq \alpha\|\psi\|_{L_2}^2,$$

as claimed.

Proof of Theorem 8.2, *the completion.* Propositions 8.5 and 8.6 imply that for λ sufficiently small, the principal G-bundle P_λ and $A_\lambda \in \mathcal{C}(P_\lambda)$ satisfy statement (i) of Theorem 8.2. Proposition 8.7 ensures that statement (ii), when applicable, is satisfied by P_λ and $A_\lambda \in \mathcal{C}(P_\lambda)$. Propositions 8.7 and 8.8 ensure that statement (iii) of Theorem 8.2 is satisfied by P_λ and $A_\lambda \in \mathcal{C}(P_\lambda)$ for all λ sufficiently small. Hence all bundles isomorphic to P_λ satisfy statements (i), (ii) and (iii) of Theorem 8.2.

9. The existence of self-dual connections

The proof of Theorem 1.2 is completed in this section. It is now a direct consequence of Theorems 3.2 and 8.2 as explained below.

From Theorem 8.2, a principal G-bundle $P \to M$ in the stated isomorphism class admits a connection $A_0 \in \mathcal{C}(P)$ with the property that $\delta(A_0) < \varepsilon_0$ as required by Theorem 3.2. Then Theorem 3.2 states that there exists $a \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$ which satisfies

$$\mathfrak{D}_{A_0} a + a\#a + P_- F_{A_0} = 0. \tag{9.1}$$

In other words, $A = A_0 + a \in \mathcal{C}(P)$ is self-dual.

Assume that the conditions of statement (ii) of Theorem 1.2 are met. We will use the measure $\mathcal{J}_m(A)$ which is defined in §8 to prove the existence of irreducible self-dual connections in $\mathcal{C}(P)$.

Suppose that A_0 satisfies the requirements of Theorem 3.2 so that $A_0 + a$ is self-dual and a satisfies (3.7). We obtain for $\sigma \in T(\rho, m)$ the apriori estimate

$$\begin{aligned} 2\int_{B_\rho(m)} * |\nabla_A \sigma|^2 &\geqslant \int_{B_\rho(m)} * |\nabla_{A_0}\sigma|^2 - 4\|\sigma\|^2_{L_4; B_\rho(m)} \|a\|^2_{L_4} \\ &\geqslant \left(\int_{B_\rho(m)} * |\nabla_{A_0}\sigma|^2\right)\left(1 - C_1\delta(A_0)^2\right) - C_1\delta(A_0)^2, \end{aligned} \tag{9.2}$$

where C_1 is independent of $A_0 \in \mathcal{C}(P)$, and the last line follows from (3.7) and Lemma 5.2.

By Theorem 8.2, a principal G-bundle $P \to M$ in the stated isomorphism class admits a connection $A_0 \in \mathcal{C}(P)$ with the following properties:

(a) $\delta(A_0) < \varepsilon_0$ as required by Theorem 3.2.

(b) There exist $\rho > 0$, $m \in M$ and $z > 0$ such that

$$z\left(1 - C_1\delta(A_0)^2\right) - C_1\delta(A_0)^2 > 0, \tag{9.3}$$

and that

$$\inf_{\sigma \in T(\rho, m)} \left(\int_{B_\rho(m)} * |\nabla_{A_0}\sigma|^2\right) > z > 0. \tag{9.4}$$

It follows from Theorem 3.2 that there exists $a \in \Gamma(\hat{\mathfrak{g}} \otimes \Lambda^1)$ such that $A = A_0 + a$ is a self-dual connection. Meanwhile, (9.2)–(9.4) ensure that there exists $m \in M$ that $\mathcal{J}_m(A) > 0$. Thus A is irreducible as well.

Statements (i) and (ii) of Theorem 1.2 have been established. Statement (iii) of the Theorem is a standard result; cf. [23], [25].

Appendix: classification of principal bundles

Let M be a compact connected 4-dimensional Riemannian manifold, and suppose that G is a compact connected semi-simple Lie group. The isomorphism classes of principal G-bundles $P \to M$ are in one-to-one correspondence with the set of homotopy classes of maps from M into the classifying space BG for G, [8], [12]. This set is denoted by $[M; BG]$

Proposition A.1. *Let G and M be as described above. Then there is a surjection*

$$\phi\colon [M; BG] \to \mathbf{Z}^l \to 0 \tag{A.1}$$

which is a bijection if G is simply connected. Here l is the number of nontrivial simple ideals which compose $\mathfrak{g}$.

Proof. The l Pontrjagin classes, [8], [12], $\{p_1^k(g)\}_{k=1}^l$ of the associated vector bundle $\hat{\mathfrak{g}} = Px_{\mathrm{Ad}_G}\mathfrak{g}$ provide a surjection

$$[M; BG] \to \mathbf{Z}^l \to 0. \tag{A.2}$$

Now assume that G is simply connected. Both M and BG are CW complexes. The manifold M has cells up to dimension 4, whereas the 4-skeleton of BG is homotopically a disjoint union of l 4-spheres, [6], [7]. Every map from M into BG is homotopic to a map of M into the 4-skeleton [26]. Thus

$$[M; S^4]^l \to [M; BG] \to 0. \tag{A.3}$$

By the Hopf classification theorem we have $[M; S^4] \approx \mathbf{Z}$, and this isomorphism is given by the degree. Hence for G simply connected, $[M; BG] \approx \mathbf{Z}^l$.

If G is not simply connected, then G has a universal covering group $p\colon \tilde{G} \to G$ which is a compact simply connected semi-simple Lie group. The covering projection p is the quotient of $\tilde{G}$ by finite subgroup Z_0 of the center $Z \subset \tilde{G}$, [20]. Thus $\Pi_1(G) \approx Z_0$. If $\tilde{P}$ is a principal $\tilde{G}$-bundle over M, the projection p induces a natural bundle map $p'\colon \tilde{P} \to p'(\tilde{P})$, and $p'(\tilde{P})$ is a principal G-bundle. The induced map on $\hat{\mathfrak{g}} = \tilde{P}x_{\mathrm{Ad}_{\tilde{G}}}\mathfrak{g}$ is a bundle isomorphism.

Proposition A.2. *Let M, G be as in Proposition A.1. Then there is a map*

$$\eta\colon [M; BG] \to H^2(M; \pi_1(G)), \tag{A.4}$$

and ϕ is an isomorphism when restricted to the kernel of η.

Proof. If P is a principal G-bundle, there is an obstruction to the existence of a principal $\tilde{G}$-bundle $\tilde{P}$ such that $p'(\tilde{P}) = P$. The obstruction is an element of $H^2(M; \Pi_1(G))$ [cf. 26], and is invariant under bundle isomorphisms, hence (A.4). It follows from Proposition A.1 that p' induces a bijection from $[M; BG]$ onto the kernel of η in $[M; BG]$. The last statement of Proposition A.2 follows from this bijection.

As we remarked earlier, the map ϕ is specified by the Pontrjagin classes $\{p_1^k(\hat{\mathfrak{g}})\}_{k=1}^l$. Let

$$
\text{(A.5)} \qquad \begin{array}{lll}
r_{\mathfrak{g}} = 4n & \text{for } \mathfrak{g} = \text{Lie algebra of} & \mathrm{SU}(n) \\
4n-2 & & \mathrm{Spin}(n) \\
4n+4 & & \mathrm{Sp}(n) \\
16 & & G_2 \\
36 & & F_4 \\
48 & & E_6 \\
72 & & E_7 \\
120 & & E_8.
\end{array}
$$

Then ϕ is given by (see [3] for the derivation)

$$
\text{(A.6)} \qquad (\text{Isomorphism class of } P) \to \left(r_{\mathfrak{g}_1}^{-1} p_1^1(\hat{\mathfrak{g}}), \cdots, r_{\mathfrak{g}_l}^{-1} p_1^l(\hat{\mathfrak{g}})\right).
$$

The Chern-Weil construction represents the characteristic class $p_1^k(\hat{\mathfrak{g}})$ by an element in $H^4_{\text{De Rham}}(M)$. For $A \in \mathcal{C}(P)$,

$$
p_1^k(\hat{\mathfrak{g}}) = -\frac{1}{4\pi^2} \int_M \mathrm{Tr}_{\mathfrak{g}_k}(F_A \wedge F_A),
$$

$$
\text{(A.7)} \qquad p_1(\hat{\mathfrak{g}}) = \sum_{k=1}^{l} p_1^l(\hat{\mathfrak{g}}) = -\frac{1}{4\pi^2} \int_M \mathrm{Tr}_{\mathfrak{g}}(F_A \wedge F_A),
$$

where the integral is independent of the choice of $A \in \mathcal{C}(P)$.

Notice that if $A \in \mathcal{C}(P)$ is self-dual, the integrand in (A.7) defines a nonnegative measure on M. Thus a necessary condition for $P \to M$ to admit a self-dual connection is that the Pontrjagin classes $\{p_1^k(\hat{\mathfrak{g}})\}_{k=1}^l$ be nonnegative.

The Riemannian curvature defines a two characteristic classes: the signature $\tau = \frac{1}{3} p_1(\Lambda^1)$ and the Euler characteristic $\chi(M)$:

$$
\text{(A.8)} \qquad p_1(\Lambda_1) = \frac{1}{4\pi^2} \int_M * \left(|\mathcal{W}_+|^2 - |\mathcal{W}_-|^2\right),
$$

while

$$
\text{(A.9)} \qquad \chi(M) = \frac{1}{8\pi^2} \int_M * \left(|\mathcal{W}_+|^2 + |\mathcal{W}_-|^2 + \frac{S^2}{6} - 2|\mathfrak{B}|^2\right).
$$

References

[1] M. F. Atiyah & R. Bott, *On the Yang-Mills equations over Riemann surfaces*, to appear.

[2] M. F. Atiyah, N. J. Hitchin, V. A. Drinfeld & Yu. I. Manin, *Construction of Instantons*, Phys. Lett. **65A** (1978) 185–187.

[3] M. F. Atiyah, N. J. Hitchin & I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London, Ser. A **362** (1978) 425–461.

[4] M. F. Atiyah & I. M. Singer. *The index of elliptic operators*: III, Ann. of Math. **87** (1968) 546–604.

[5] M. F. Atiyah & R. S. Ward, *Instantons and algebraic geometry*, Comm. Math. Phys. **55** (1977) 117–124.

[6] A. Borel & F. Hirzebruch, *Characteristic classes and homogeneous spaces*. II, Amer. J. Math. **81** (1959) 315–382.

[7] R. Bott, *An application of the Morse theory to the topology of Lie-groups*, Bull. Soc. Math. France **84** (1956) 251–281.

[8] ______, *Lectures on algebraic and differential topology*, Lecture Notes in Math. Vol. 279, Springer, Berlin, 1972.

[9] J. P. Bourguignon, *Formules de Weitzenböck en dimension* 4, Séminaire A. Besse sur la Geométrie Riemannienne de dimension 4, Exposé No. 16, 1979.

[10] J. P. Bourguignon & H. Karcher, *Curvature operators*: *pinching estimates and geometric examples*, Ann. Sci. École Norm. Sup. **11** (1978) 71–92.

[11] J. P. Bourguignon & H. B. Lawson, Jr., *Stability and isolation phenomena for Yang-Mills fields*, Comm. Math. Phys. **79** (1980) 189–230.

[12] E. H. Brown, Jr., *Abstract homotopy theory*, Trans. Amer. Math. Soc. **119** (1965) 79–85.

[13] N. H. Christ, E. J. Weinberg & N. K. Stanton, *General self-dual Yang-Mills solutions*, Phys. Rev. D **18** (1978) 2013–2025.

[14] V. A. Drinfeld & Yu. I. Manin, *A description of Instantons*, Comm. Math. Phys. **63** (1978) 177–192.

[15] R. Hartshorne, *Stable vector bundles and instantons*, Comm. Math. Phys. **59** (1978) 1–15.

[16] A. Jaffe & C. H. Taubes, *Vortices and monopoles*, Birkhauser, Boston, 1980.

[17] S. Kobayashi & K. Nomizu, *Foundations of differential geometry*, Vol. I, Interscience, New York, 1963.

[18] C. B. Morrey, *Multiple integrals in the calculus of variations*, Springer, New York, 1966.

[19] R. S. Palais, *Foundation of global analysis*, Benjamin, New York, 1968.

[20] L. Pontrjagin, *Topological groups*, Princeton University Press, Princeton, 1946.

[21] C. H. Taubes, *Arbitrary N-vortex solutions to the first order Ginzburg-Landau equations*, Comm. Math. Phys. **72** (1980) 277–292.

[22] ______, *The existence of multi-monopole solutions to the non-Abelian*, *Yang-Mills-Higgs equations for arbitrary simple gauge groups*, Comm. Math. Phys. **80** (1981) 343–367.

[23] K. K. Uhlenbeck, *Connections with L_p bounds on curvature*, to appear in Comm. Math. Phys. **83** (1982) 31–42.

[24] ______, *Variational problems for gauge fields*, to be published.

[25] ______, *Removable singularities in Yang-Mills fields*, to appear in Comm. Math. Phys. **83** (1982) 11–29.

[26] G. W. Whitehead, *Homotopy theory*, M. I. T. Press, Cambridge, Massachusetts, 1966.

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